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Masatsugu Urabe (Eds.)

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Japanese Conference, JCDCG 2000
Tokyo, Japan, November 22-25, 2000
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Preface

The Japan Conference on Discrete and Computational Geometry (JCDCG) has been held annually since 1997. One of the goals of this conference series is to bring together Japanese researchers from both academia and industry with researchers in these fields from abroad to share their recent results.

JCDCG 2000 was held 22–25 November 2000 at Tokai University in Tokyo in conjunction with the celebration of World Mathematics Year 2000. A total of 120 participants from 20 countries attended. This volume consists of the papers presented at JCDCG 2000, which have been refereed and revised. Some papers which appear in short form in this volume also appear in fuller expanded versions in journals dedicated to computational geometry.

The organizers of the conference thank the principal speakers for their interest and support: Imre Barany, Erik D. Demaine, Greg N. Fredrickson, Gyula Karolyi, Naoki Katoh, David Kirkpatrick, Joseph O'Rourke, Janos Pach, Jozsef Solymosi, William Steiger, Jorge Urrutia, and Allan Wilks. They thank the major sponsors for their generous contribution: The Research Institute of Educational Development of Tokai University, the Ministry of Education of Japan (for the grant-in-aid to A. Saito (A):10304008), and Tokai Education Instruments Co., Ltd.

April 2001

The Editors,
Jin Akiyama
Mikio Kano
Masatsugu Urabe

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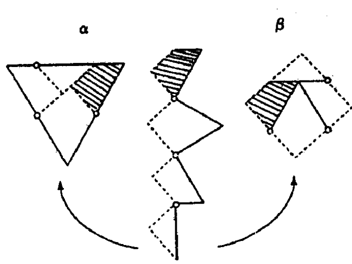

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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 1-30, 2001.

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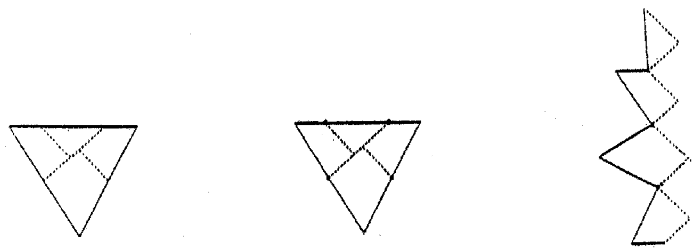

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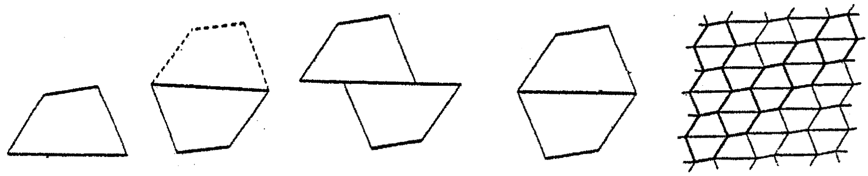

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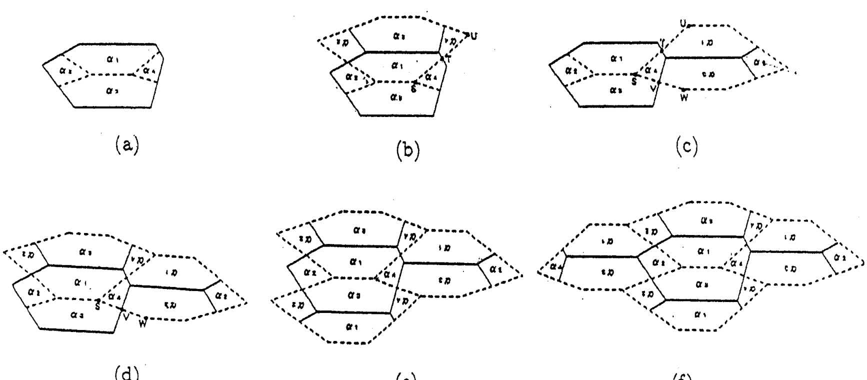

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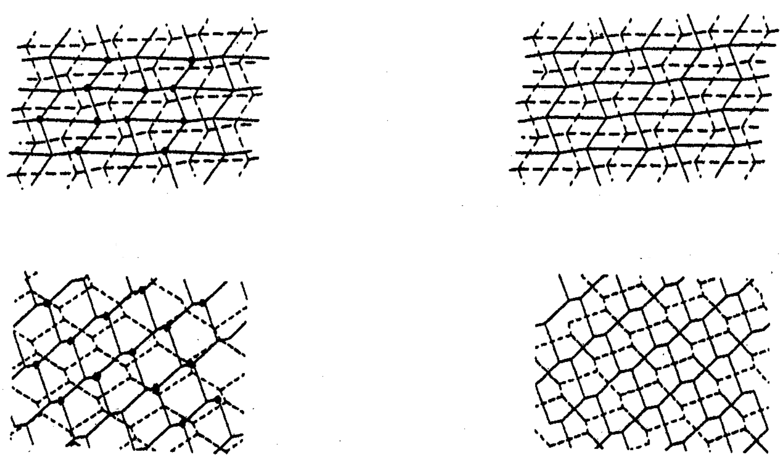

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Chapter 3. Verious Results of General Dudeney Dissections of Polygons

The results can be classified into two, namely general Dudeney dissections and congruent Dudeney dissections. A general Dudeney dissection of a polygon α produces a polygon β, which is not congruent to α. A *congruent Dudeney dissection* of a polygon α produces α itself. In this section, we introduced various results on general Dudeney dissections which are discussed in the paper [2]. It provides procedures for constructing the dissections shown below.

Every quadrilateral has a Dudeney dissection to a parallelogram (Figure 3.1).

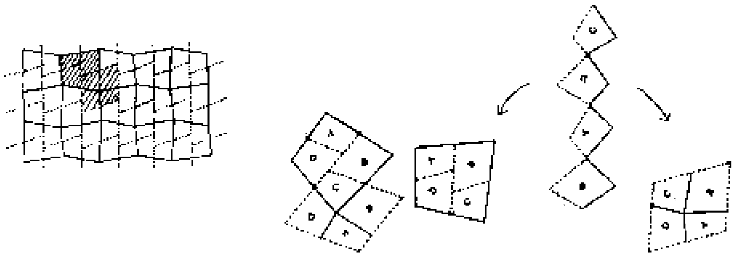

Figure 3.1

Every triangle has infinitely many Dudeney dissections to parallelograms (Figure 3.2).

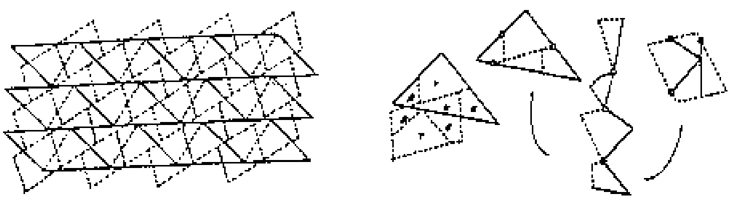

Figure 3.2

Every parallel hexagon has a Dudeney dissection to a trapezoid (Figure 3.3).

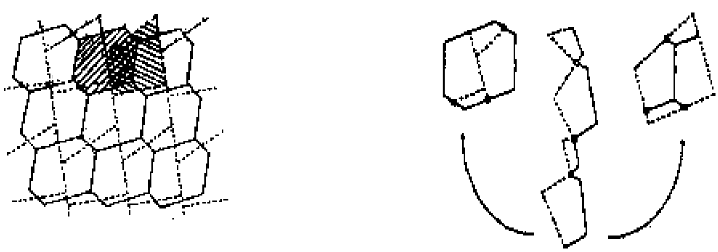

Figure 3.3

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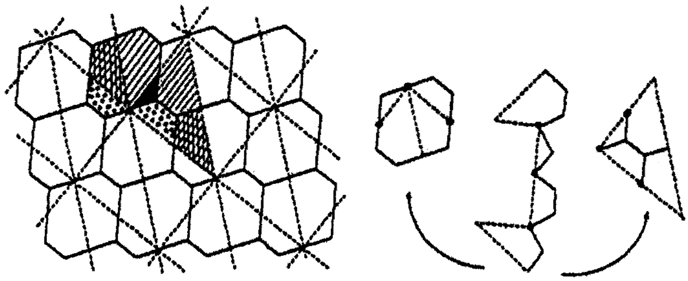

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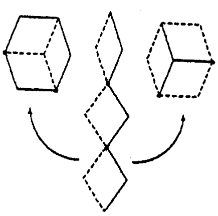

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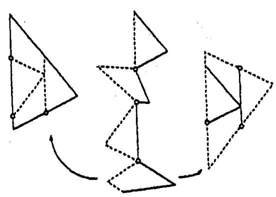

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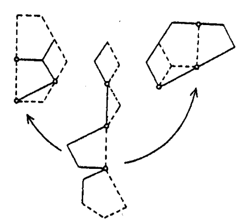

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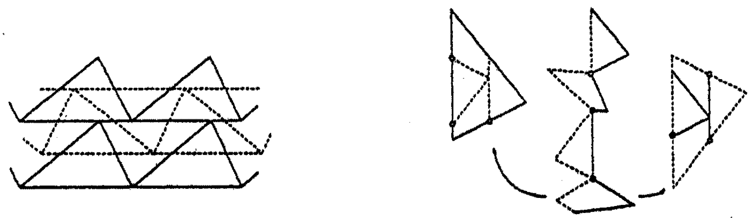

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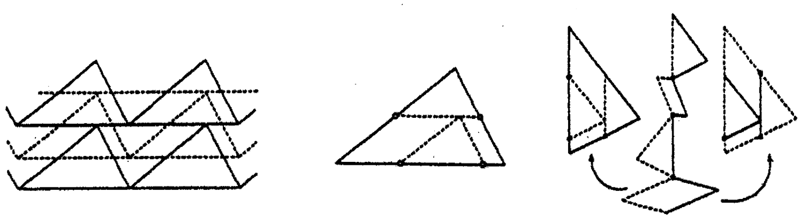

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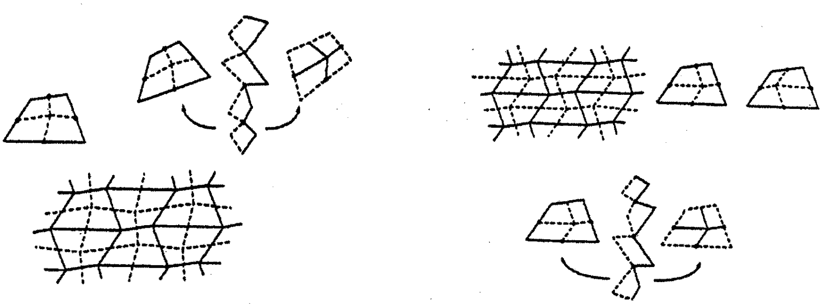

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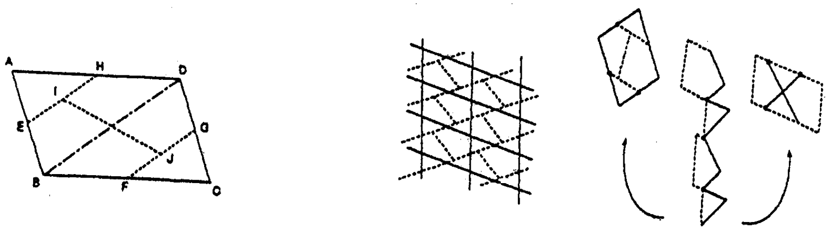

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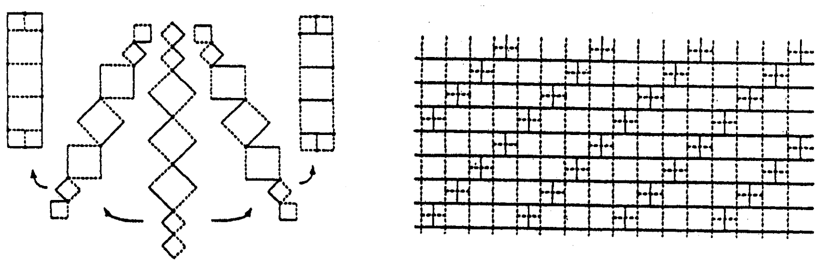

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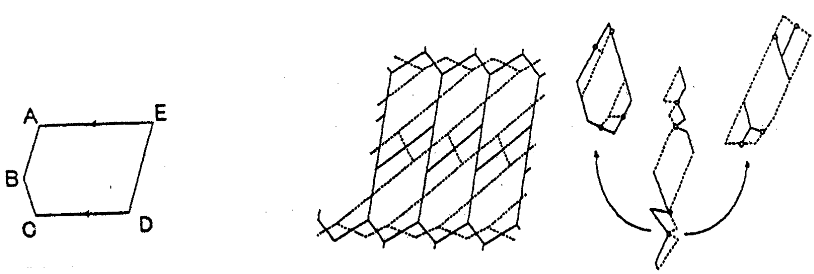

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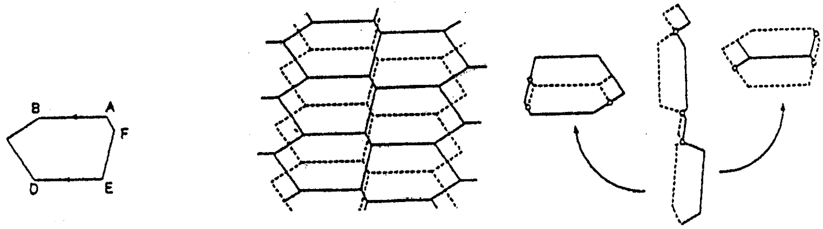

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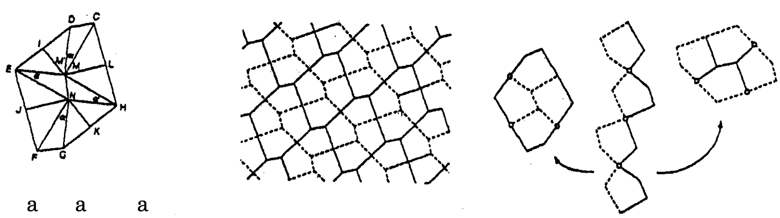

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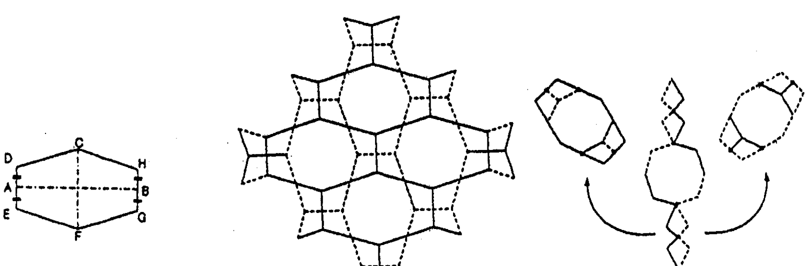

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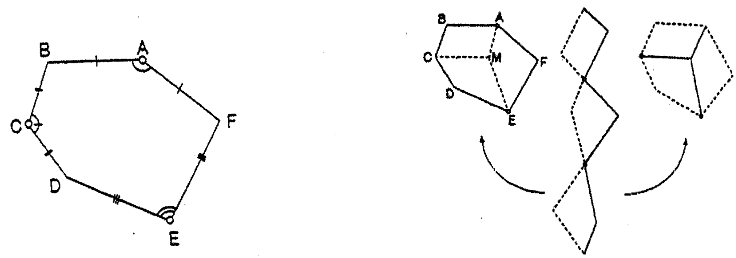

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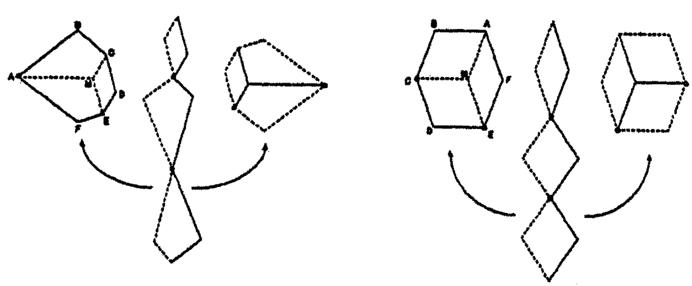

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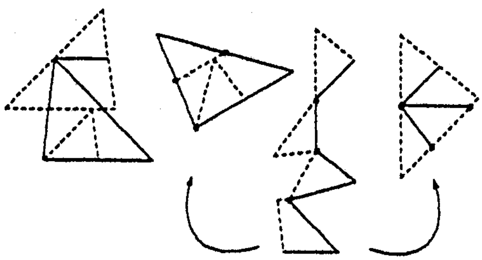

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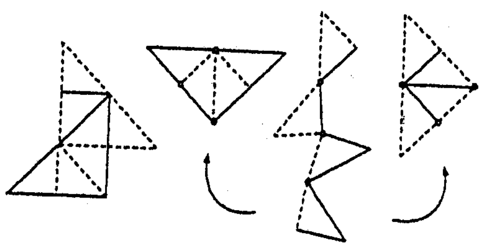

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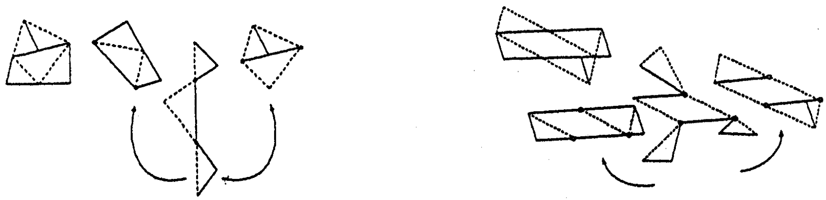

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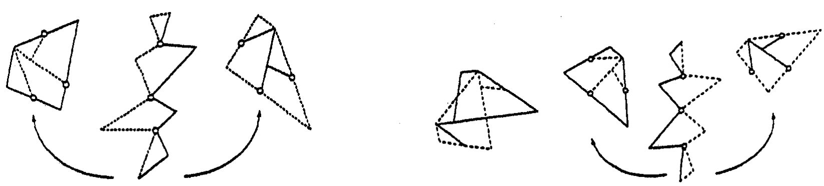

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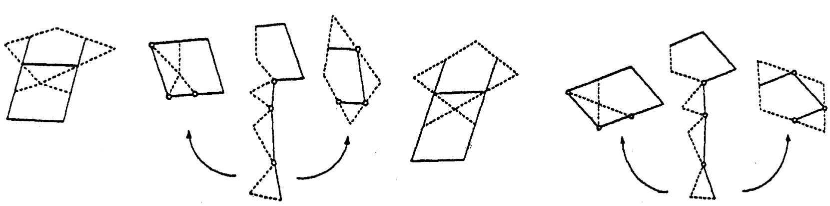

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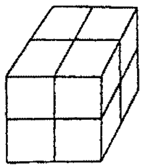
I e c o ew pe o D de e d ec o o pol ed o o e co e o e d ewe v e e o D de e d ec o o ol d oweve o w o d cove ew pe o c d ec o o eed d c pl ed pp o c p z d w o e c w ll o eld ood e l ll e w e e o o ld e plo o e c ? Yo o ld e o e c l p c ple l be d D de e d ec o o ol d o e c lo ve o we cceeded d cove e e e le o d e e



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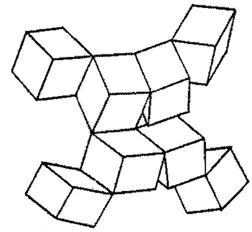

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e c l p c ple w c ove D de e d ec o o pol ed o I we ollow e c o e e e c l p c ple e c c e c ll o w ll c c ll o ce l e l o D de e pol ed o o ol d b ed o p c ple I be po ble e e e ll o e e c l p c ple be de e e o b p e e we do o ow w e e c p c ple e we ve dec ded o cl D de e d ec o o ol d o e ollow pe

p ce ll pe

e ed pe e ed pe 9 b e ed pe 7
c M l l e ed pe

M o e pe

Cl p pe

e e o o e c o e e pe c w ll beco e cle we p oceed o ve e e ec de l D de e d ec o o ol d

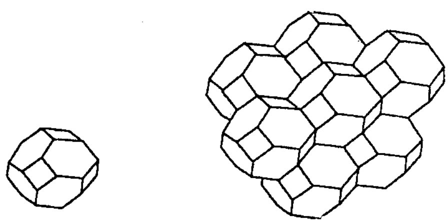
te e i t e D e e i e tio e i o itio o two te e tio

ol d c lled *i g id* o *r* p ce c be lled w o v p o ove l p b cop e o e ol d e e e c ol d c c be ec l bloc pec l d o e ed o l p e e l ed o c ed oc ed o F o b c dodec ed o F d o o e e o o de l

B p ce ll pe we e e o D de e d ec o o ol d co c ed b cleve e o p ope e o p ce ll ol d w c e e l z o o e pe po o o wo l ppl ed o pl c e e e pl e od we c ll e od pe po o o wo e ell o o e co c e el b wo e ple o e e l ed o d o b c dodec ed o

(d n d t n b t n d n nd t n b k

we de c bed bove o c pl ce e e l ed o o e ze e o o e co b e e e p ce ppo e o e o j wo o e e ed o d c e F e B co ec o ve ce l o e lowe po o o e c o e e ed o b do ed l e o o wo q e d co ec e e q e w o ve c l do ed l e o ee ec l bloc w ll e e e o e wo ed o F e b Yo d cove b epe e e p oce o e c ve c ll dj ce p o ed o o e d p w e o e co b o e p ce b ec l bloc F o e w e c ec l bloc w co c ed cle e el ve po o o e e l ed o d ec l bloc e co p ce d e vol e o ed o d o ec l bloc e e



s m r r 4 r ns

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p ce ll pe D de e d ec o o ol d e co c ed b ed o p ope o p ce ll ol d ppo e o d w e ell o o e p ce b e o e e l ed o d d w o e b e o ec l bloc d pe po e e e d w Yo e d ec e e l ed o o p ece b e o e pl e o e e e o ce o ec l bloc e ec e ed o e o ob D de e d ec

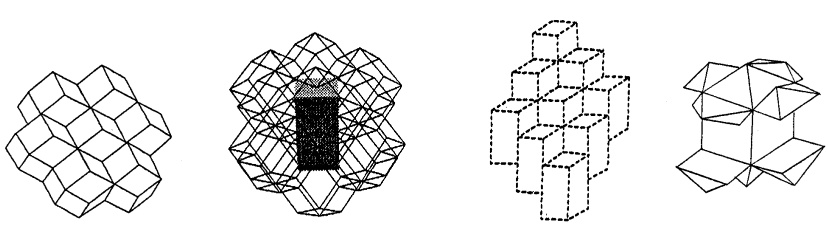
o o e ed o o e ec l bloc F e c ll e
c I o oll e p ece o d ec o o e e e e l ed o
w le o oll o e p ece o e e ec l bloc e c
deed w ppe o e o le obv o o e w e e l
ed o d e ec l bloc o e co b e p ce

S p p t n t t t n t p

I o pe po e e ell o o e p ce b o e o p ce ll ol d
o o e ell o o e p ce b o e c ol d d o w e eb
eep el ve po o o e e ol d co d e d ec o e o e
ol d o o e d b e pl e o e e o ce o e ol d o e o e
d e ec e o e o w ll ob p ce ll pe D de e
d ec o be wee e e ol d e e p ze e e e pe po o o
e ell o o be do e d o l po eve
o e e w e co e po d ve ce d de o ol d e p o e e b
ed pe d o ed

(b d n d t n b t n b d d d n nd t n b k

o e e ple o p ce ll pe D de e d ec o o ol d le
e pl D de e d ec o be wee o b c dodec ed o d ec
l bloc b ppl o pe po o o wo e ell o o e p ce
o b c dodec ed o p ce ll ol d e c ll p e p ce w
o p d o ove l p F e b cop e o ol d B
p c po loc ed e e po o o e c o e co e po d ed e
o e ce o e e dodec ed o d co ec e b do ed l e
ow F e b we ob ec l bloc e e ec l bloc
l o o e co b e p ce F e c d b d ec o e o e
o b c dodec ed o o 9 p ece b e o e pl e co p e e
e o ce o e ec l bloc e ec e o b c dodec ed o
we ob F e d D de e d ec o be wee o b c dodec
ed o d ec l bloc B e o pe po o o e ell o we
c ob p l D de e d ec o be wee wo d o p ce
ll ol d



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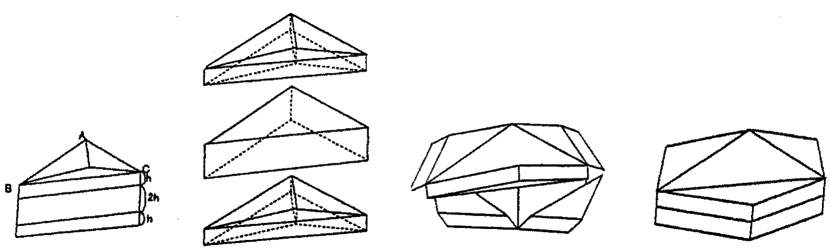
te L e e t e D e e i e tio o o i
– d t nt t p n d n d t n –

ol d c l o ec l p w c ve e pe o
p ll e c lled *ri* e l e l de o p e pe pe d c l o
e op d e bo o c lled *righ ri* B l e ed pe D de e
d ec o o ol d we e e o ce cl o D de e d ec o o
p e e e wo e e ll d e e D de e d ec o o p
o e l e ed pe d o e l e ed pe O b o e e wo
c e we c p od ce D de e d ec o o ce pe o p w c c
be ob ed b l c e o l e o b e w
l e ed pe

(d t p d n t n S d

cl o d ec o e c c e c o v wo pl e p llel
o e op d bo o o e p l c e p o ee l e e e we
e p ze we e e e o o e p c l cl o D de e d ec o
o p e e e o e cl c lled l e ed pe d ec o
o p



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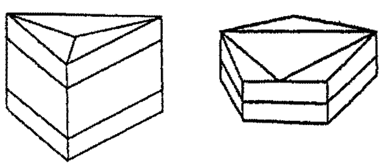
t d d t p d n d t n d

e e e ple c e o l p d e pl
D de e d ec o be wee l d e o l p I
F e 7 l p o e w c p o ed o
ee l e o l p b e o e wo pl e p llel o
e op d bo o o e p e e e d e d l e ve
e w le e eco d l e e e e be c be
b po ve be e le ve e eco d l e p c w le e
d d l e p e d ec ed e o ee lle l
p e c b e o ee pl e pe pe d c l o e op o e l
p d e ec e op le lo e ee l e e e co ec
po P b l c o e o e e o o e le d e ve ce
B d C o le e ee l e ow F e 7 b ll e

e o d ec o e e c e ee p ece o e l e d
o e o e d l e o e eco d l e p b l e b ed
pe lo e co o ed e ow F e 7 c d e pp
e e lle l p ove e l ed ed e we ob e o l
p o e ow F e 7 d e o ce b
p oced e we e d p w v e op d e bo o ce d e l e l
ce o e e l e o l p co o e c o ec o o e
d ec o o e o l l p e e eve l
p ope d e e o e we ve D de e d ec o be wee l
p o e d e o l p o e p ov de
e ple e ple o l e ed pe D de e d ec o o ol d F e 7
ll e D de e d ec o be wee l p o e
d pe o l p o e w le F e 7 ow D de e
d ec o o e e pe be wee ec l p o e d
q d l e l p o e



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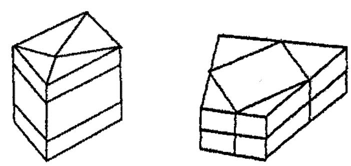

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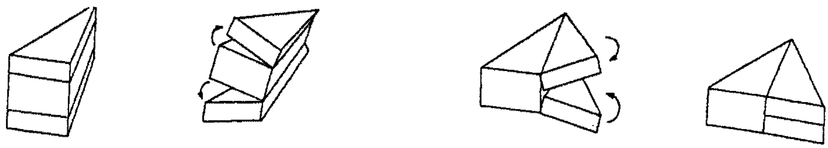
d t n t t p n d n d t n

o e o e p l o we c e e d e l e ed pe D de e
d ec o d c ed bove o o e e e l l e ed pe D de e d ec o o
ol d I o de o e pl e de volved le e ve ple e ple
o l p w o e op d bo o le ow
F e 7 e e o p d p o ed o ee
l e o l p w e d e pec vel e co ec e
d e d l e o e eco d l e b l e b ed pe
lo e l e o d ec o ppe o e e l e l de o e o l
p d e p e d e d l e p ove e l ed ed e
ow F e 7 b o o e e ow F e 7 c we e d
p w l p o e ow F e 7 d
e op d bo o o e e l p e o cele le d e
e de p o e c o ec o o e d ec o o e o l p oweve
e c o e ee l e l de o e e l p ll de p o l e l
ce o e o l l p o le e o w o co ve
e e l e l de o o e co o c o ec o o e d ec o o e
o l p I o de o c eve ce o co de pl co e
D de e d ec o o e o cele le ow F e 7 5 I w



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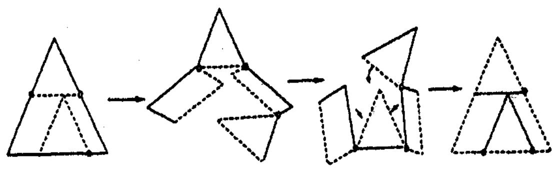
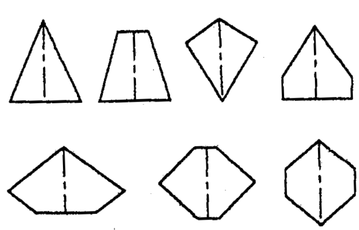


F 5

we ob D de e d ec o o l p ow F e 7 5 w c ob ed b d e e e o e o e d c ed e l e

n t n t d t d

Fo e e l l e ed pe D de e d ec o o ol d d c ed bove e op d bo o ce o e p we e pped ove o e o e ed e o e e l e l de o e o l p o o pol o w c e e c w e pec o e e ed e e e o e we c c eve pl D de e d ec o o pol o e we c ob D de e d ec o o e o l ol d Pol o ow F e 7 cl d e o cele le o F e 7 d ll o w c e e c w e pec o l e e e ve c pl D de e d ec o D ed l e e e p c e ow e e pol o e ob ed b pp p ece ove e e l e B ppl e od we c ob d e e d o l e ed pe D de e d ec o o ol d F e o e ow o c ppl b c de w o e cleve p l o o ob D de e d ec o o p w c e o p b l ed o ew



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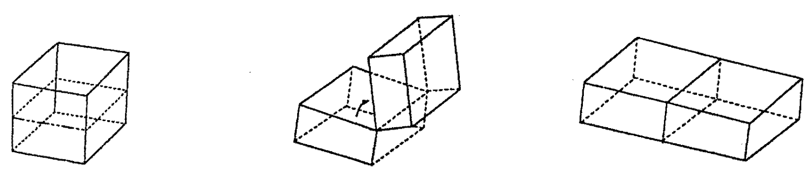
(b d t p d n t n S d

l e ed pe D de e d ec o o ol d e e o D de e d ec o o p w ec e o l e ed pe D de e d ec o oweve l e ed pe d ec o ve c c e c o d ec p o wo l e b e o le pl e p llel o e op d e bo o o e p w e e l e ed pe d ec o d c ed e l e d ec ed p o ee l e I o de o e pl co c e el e ec volved le e e ple o c be w c e o v l ze ppo e we e c be d co de w o co ve o c be B co e o e c be b e o D de e d ec o o l e ed pe e p o F e 7 7 e c be b l c b e o pl e p llel o e op o e c be o ob wo co e q e bloc e e p o e e e e q e bloc b l lo o e o e ed e o e c o ec o b ed pe F e 7 7 b d b pp e ppe bloc ove e l ed ed e we e ec l bloc o F e 7 7 c ec l bloc w d le d e o d $\frac{}{2}$ e le o e c ed e o e l c be eq l



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l (ee ei f e l

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f e i f e
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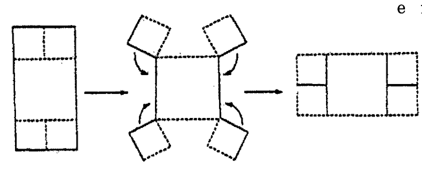

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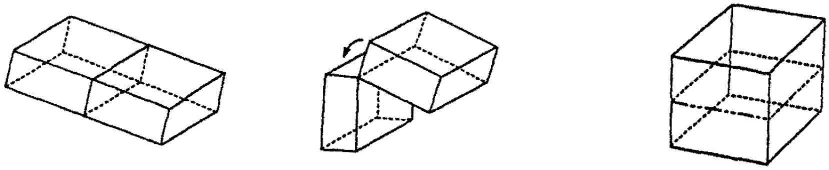
I we ob e ve e e e o ce o ec l bloc we o ce e op ce co o e c o ec o o e d ec o o e c be b e l e l ce d e bo o ce e e e o ce o e c be o le e o w o co ve e l e l ce o e ec l bloc o ce co o e c o ec o o e d ec o o e c be Fo p po e we c e pl D de e d ec o o e op ec le o e bloc F e 7 7 d ll e p oced e e ow o e c e o e bo o ce b c be do e e l b o o e p oced e d c ed F e 7 b c I w D de e d ec o o e c be o e c be B cco pl ed



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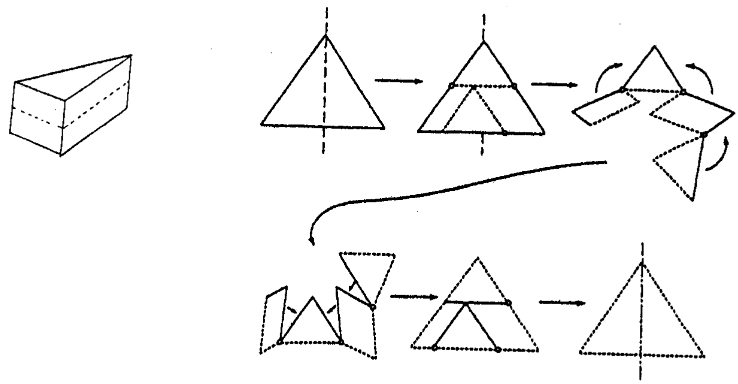
n t n t d t d

e o e D de e d ec o o ol d o pe d c ed bove co
o l c e c be o wo l e o e ec l bloc w l e
e ollow w pl D de e d ec o o e op ce o e bloc
d ll c o e l o e ec l bloc o op o e o e l
o e o e e wo l e e p oce o l c e c be d o c
p ece o e o e wo l e e bo q e ple d e e o e e e o
l e ed pe D de e d ec o l e pe o pl D de e d ec
o o e op ce o e ec l bloc c c e c p ope o
pl D de e d ec o de l w wo pol o e c w
e pec o ce e p oble o d D de e d ec o o ol d
ed ced o d pl D de e d ec o beco e c ple
e e ple o pe o D de e d ec o o ol d we ve wo
o e e ple o D de e d ec o be wee wo co e ol d F e 7 9
ll e co e D de e d ec o o do ble c e p v
e op d bo o ce o eq l e l le w le F e 7 ow
co e D de e d ec o o do ble c e p v e op
d bo o ce o o cele pezo d l ce e loo e ple
b e e c ll q e c e e c co c o e e ple o
l e ed pe D de e d ec o o l ol d

(t d p d n t n S d

o e p oced e volved wo l e ed o ee l e ed
pe D de e d ec o o ol d d c ed e l e we c ow D de e
d ec o o ce pe o p c be ob ed b l c e o l e
o b e ve o ve p w e op d bo o co
o pol o o e pec l d o de o co de l e ed d ec o o
b e ple e ple ve b e c e o ec l p
ce c e ll e well e e e l e e o l l e ed D de e
d ec o d c be e pl ed q e pl we w ll co de ec l
p e eq el



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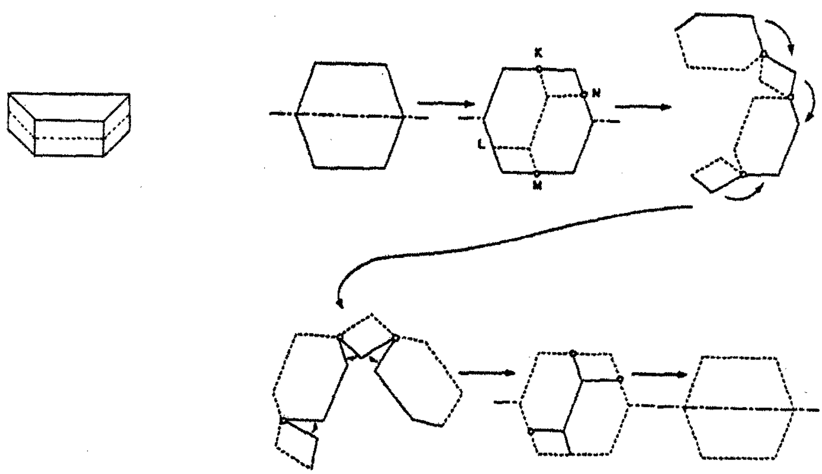
d e d w do ble c e p v e op ce o
eq l e l le



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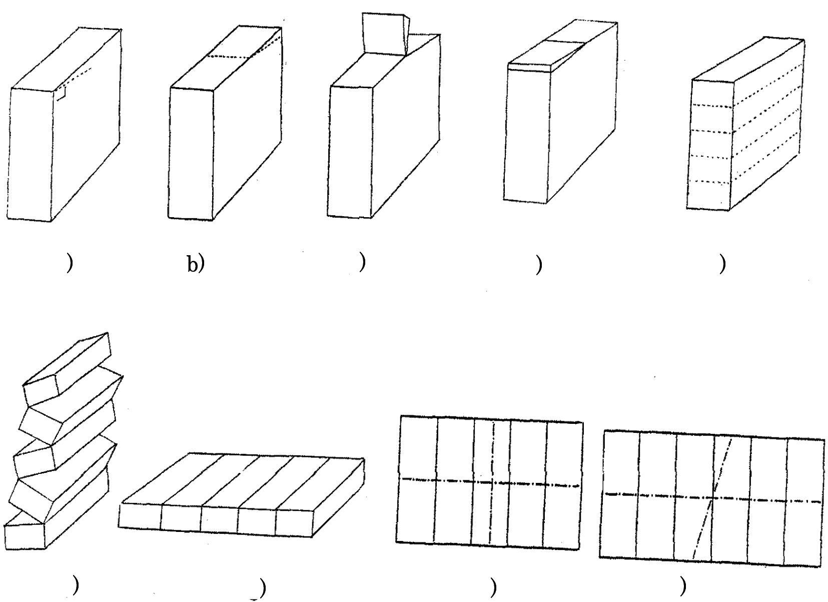
d e d w do ble c e p v e op ce o
o cele pezo d

n x p t d d n d t n t n p

e co de p v ec l op d bo o ce F e 7 p oweve o p b l l l w ll pl c ole e l e ed pe D de e d ec o o ol d e e o o co e o e c ece o o o ope o o c e e op d bo o ce o e p o ce co o c o ec o o e d ec o be o e we pe o l c o l e ope o c be e ec ed q e e l oweve Fo e ple we w o c e e op ce o e ce co o c o ec o we d v de e op ec le o wo eq l p b c lo e do ed l e d c ed F e 7 b d c o l wed e be e o e l o e ec le d p ove e do ed l e d l e o e o e l o e ec le F e 7 c d 7 d ll e p oced e e ee e do ed l e o e op ce pl e ole o e p oce I we pe o e e ope o o e bo o ce we ob ec l p w bo op d bo o co o c o ec o o e d ec o I e l wed e ed e p oced e bove le o de ce e e ec l p ob ed e ope o w ll beco e p e ed o w ec l p w l o de o c eve o l el o e d p w ec l p I w we e ec l p ow F e 7 e w op d bo o ce co o ce o c o ec o e ow d ec p o ce o ve l e



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F e 7 e d p e d o b old d ec ed p ece ove e l e o d ec o ppe o e l e l de o e p F e 7 ll e p oce d F e 7 ow e e l ec l bloc ec l bloc e w d ve e e w d o e o l ec l p d e e o e o e p I e o c ec e op d e bo o ce o ec l bloc co o e ce o c o ec o o d ec o e o ce o e o l ec l p e o ec l bloc o l o e o l e l ce e e o e D de e d ec o o e o l ec l p w ll be ob ed we pe o pl D de e d ec o o e op ce o e ec l bloc I pe o c pl d ec o we o ld be c e l o o e e o l l e o d ec o o e ve l e ppe o e op ce o e bloc Bec e o e c o e e b le d c o c oo co ec pl D de e d ec o depe d o e be o l e be eve o odd F e 7 ve o e e ple o e pl d ec o o c e o odd be o l e d F e 7 ow o e e ple o eve be ed l e c e e o pl d ec o e d c ed b d ed l e w do e e e

l o we d c ed o l e c e o p w ec l op d bo o ce bove e e e be o c e o e p w d e e pe o pol o o op d bo o o w c l l e ed pe D de e d ec o c be pe o ed oweve w e we old e p e l c o l e we ve o old o o l o e d ec o w e c e e e ple bove b o wo d ec o o le o d p d dow o e c e o old le we ve o old z z edl

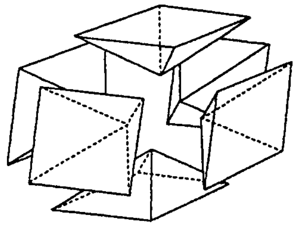
te Mi o i e t e D e e i e tio o o i

B o e pe D de e d ec o o ol d we e D de e d ec o o ol d ob ed b cleve e o e p ope e o e ec o b o I o de o e pl o e co c e el w pe o D de e d ec o l e le e e ple o c be c be v l zed cle l

t d t p d n d t n b

ppo e we co de pl e p o e ce e d o e o e ed e o c be pl e co o e ed e o e c be ed e c ll c o e ce e o e ed e e e o e we co de ll e pl e co e ce e d o e o e welve ed e o e c be o ob d e e pl e I we d ec e c be b e o e e pl e we e F e co e q e p d

e we co de e o e c o e e p d ob ed b e ec b o ed o b e w c o e o e ce o e c be e we ob oc ed o o ed b e c co b o o p d d o e dw c e o o bove d below F e ll e e w wo q e p d o e be e o e o e o e pp o c e c o e o bove d below d F e b ow e oc ed o ob ed w e e e wo p d co e o e e e l we



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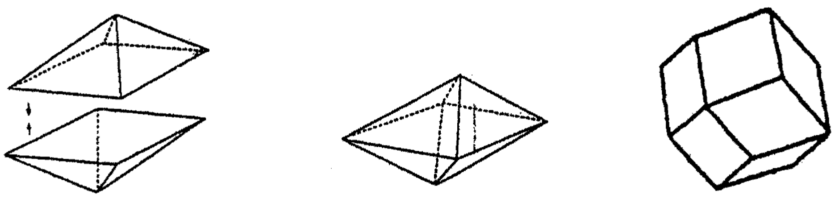

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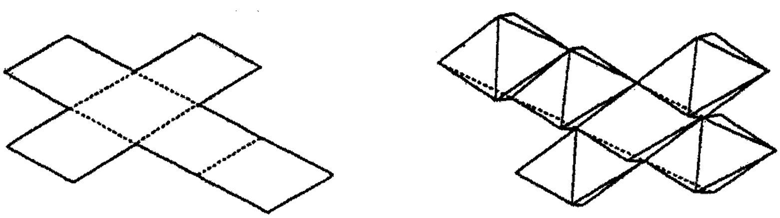

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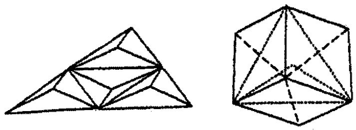
ob ol d v q e p d c ed o e c o e ce o e c be d ol d o b c dodec ed o v ve ce e op ve ce o e c ed q e p d F e c ll e e o b c dodec ed o ob ed w

I we e p oced e o ob o e pe co e D de e d ec o o o b c dodec ed o ed ve o o o b c dodec e d o ppo e we d w develop e o c be F e b d we c o e c o e q e wo q e p d o F e o e o e op de d e o e o e bo o de o e q e o e oc ed o F e b e e l e o e e ow F e b e dj ce oc ed o b ed pe lo e ed e d c ed b do ed l e o dj ce q e o e develop e B old e e oc e d o b pp e w d we e o b c dodec ed o d b old e b pp e o w d we l o e o b c dodec ed o o e e

pe e we ve co e D de e d ec o o o b c
dodec ed o I we e ol d d e e o q e p d we c ob
D de e d ec o o ol d c dec ed o well



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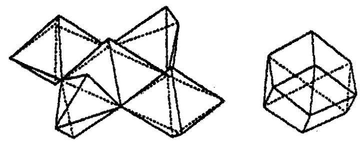


F 5

e c c e c e e o D de e d ec o o ol d o pe e
c b c ol d c be e c e bove ve o w d b
e ec e c o e q e p d l de e c be b e o
o loc ed o bo o ce w c o e o e ce o e c be we
e q e p d w c e o e o e e o p d
c ed o e c be o o de e e o w e pe o D de e d ec o
ob ed w c lled o e pe cle e e l o
pl ce o c be e ed o d l p e b c ol d o
e ope o e e o w e e ol d c be ed e b c ol d l e
e c o e le o ed b wo b e e o ce o c
ol d lw le 9 de ee I we o o e e p oced e
bove b w e ed o o w l p o e ple we
ob D de e d ec o o dodec ed o o F e o o F e 5 e
pec vel I ve c e ll e ec volved o e pe
D de e d ec o o ol d we l o ob e o c D de e d ec o o
ol d w c volve o e ec ed e

te i e D e e Di e tio o oi

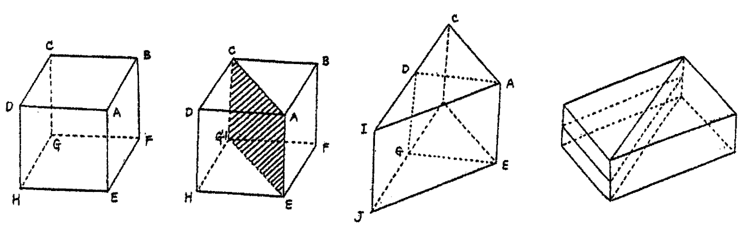
B cl p pe D de e d ec o o ol d we e o e D de e
d ec o w c volve le o e ppl c o o e cl p ope o
de c bed below Bec e o de o cl o D de e d ec o
o d jo o e cl e o D de e d ec o o ol d d c ed e l e
p ce ll pe wo l e ed pe ee l e ed pe d o e pe
oweve ce e e e D de e d ec o o ol d w c belo o e
cl p pe b o o e o e o e pe e l o e d
co o e cl

e ow e pl w cl p ope o e ol d pol ed o
o w c we w o pe o D de e d ec o d p c o o e o e ed e
I e wo ce w c e pol o o e pol ed o b ed e e
e c w e pec o ed e e we c cl p e e wo ce o e e
b l c e pol ed o b e o pl e o ed e jo e

wo ce w e lo e ed e d e ove o e l ced l o e pol ed o ove e ed e e ll ope o de c bed be *h* *i* *g* *r* *i*

(b

e loo ve ple e ple volve co e D de e d ec o o c be o F e 9 e de e e op ce q e o e c be BCD d e bo o q e F d p c o e ed e wo q e ce FB d D b e ed e e e c w e pec o e ed e d e e o e e cl p ope o c be pe o ed o e e q e ce e l ce e c be o wo p d c ed F e 9 b b e o e pl e o o e ee po d C e p e o e ed e d o e o e l p ob ed b e l c ove e ed e o o cl p o e e e wo q e ce F e 9 c ll e e o co e o ope o ve l p w o cele le o e op d bo o ce I we ppl ee l e ed pe d ec o o e l e l ce co e ed e CI we e ec l bloc F e 9 d w e e l o e o l c be d e e o e op ec le w ce o e q e ce o e c be o e e op d bo o ce o e e l ec l bloc e de p w e ce o e c o ec o ob ed b e l c o e c be e e o e we ppl pl D de e d ec o o e op ec le o e ec l bloc e we cco pl co e D de e d ec o o e c be

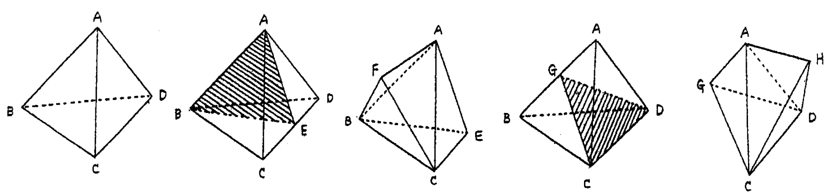


d

F

(b x d n

l o e e ple bove d c ed F e 9 – d l o v l e e e ple ve pec c l D de e d ec o o e ed o o o e e ed o Fo e ple we o w pec l e ed o BCD F e 9 O e ed o o l ce co o wo p le BC d BD d le CD d CDB e le BC d BD e p e e c w e pec o e ed e B w le CD d CDB e eco d p e e c w e pec o e ed e CD Co eq e l C D BC BD old d e p ed le e co e d e e o cele le I we ow c oo e b po o e ed e CD o e e ed o d c ed F e 9 b d d ec e e ed o b e o e pl e p o e ee po B d e we c cl p o e e e wo l ce BC d BD b o e o e wo l ced po o ove e ed ed e B F e 9 c ll e e e ed o ob ed b p oced e I e e e we c cl p o e e e wo l ce CD d CDB b b po o e ed e B d d ec e e ed o b e o e pl e o o e ee po C D d d o e o e l ced po o ove e ed ed e CD F e 9 e ow e e l e ed o I we ppo e we we e ve e e ed o o F e 9 c o w d o o e e ed o o F e 9 e v e p oced e ll ed F e 9 b d 9 d e we ob D de e d ec o o e ed o o o e o e p c l e ple o cl p pe D de e d ec o o ol d d doe o belo o o e pe p ce ll wo l e ed ee l e ed o o e d c ed e l e



d

F

kn d nt e o e F ede c o o v l ble e o M Jo z d Y j I o o e ce o o e p ep o o p pe

Re e e e

y : l ss n n bl p bl s l s *g M*
s d s s x n ss) 0–
y : n y ss n p ly ns *s d*
G y l s) s n C p n
3) p n l 000) –
y : C n n n y ss ns p ly ns – ll
n p n s s p ly n– s b *P* *t* *d*
G *d* *s g s d*
s s M
y : C n n n y ss ns p ly ns – ll
n p n s n s s p ly n– *P M s d*
n n s y 000) 5– 5

5 y : C n n n y ss ns p ly ns – n p n s n b s n s s p ly n– n p n s) *s s s d v k v* n 000)
6 y : n y ss ns p ly ns – p fil n s s l s– n p n s) *s s s d k v* 000) –
y : n y ss ns p ly ns –T l y yp – n p n s) *s s s d k v* 000) – 6
y : n y ss ns p ly ns – yp – n p n s) *s s s d k v* 000) – 5
y : n y ss ns p ly ns –T l y yp – n p n s) *s s s d k v* 000) 6–5
0 y : n y ss ns p ly ns – l l y yp – n p n s) *s s s d k v* 000) 5 –6
y : n y ss ns p ly ns –Cl sp n yp – n p n s) *s s s d k v* 000) 6 –66
l y ns : *H s d P* ns n ns T nsl by l n)
ly F : *v* s s ly n : Typ s C ll p s p n l n n n))
ss p : bl s *v d M* 6) –
5 C x : *d s G* l y ns 65)
6 n : ¨b n n l *v d G s s d W s s s z G¨ g M s P s k s ss* 00) 5– 5 b s n ly n *M s* 0) 65–
n y : *P zz s* T s ls n ns 0)
F s n : *ss s P & C* b n s y ss)
n : s n n j b l b n n l n l n l n n F n n s lb n ¨ ¨ *d d g d M k s*) – n T
0 ¨nb p C : T l n n n l s *MS* **3** 0) 5 –
lb : s bl *v d G s s d W s s s z G¨ g M s P s k s ss* 00) bs n ly n *M S* 0 0) –
l n : *M G d* s n n n l)
n n : *G ss s* n s n C p ny 6)
y l : C n ns n ss s s s n s p l l n s p ly s l sp l n s ns ns *M H v* 65) – 0
5 ll :) *s G*) ll F s s x b s l s ppl n by n l y)

s l s s W h s

J o F d[2] o o
Jo e U [3] d C lo Z o C [3]

s ns n l l p n T n s y T y
b y T w m n we ne
aka - ka a
l ns n n n s n s y 5
5 6 k a - ka-ken a
ns s n s n l n x l
n C n fi C x C n s x F C 0 5 0
C am ra ma em nam mx rr a ma em nam mx

a . s n ps y y ns n s s s n l ny n s s s s n s n p s lly s ns n s s s n s p p s y s n s ns b n l ss n s ny n l n s y l s l p ll p y T s s ll b ll *v s s g d v s* n l s l s n n l b s n b n n n l b s n s p 6 l s

t o tio

co o dev ce ed o e e l q d J p e e o e o e e o w e bo w q e b e o e d e B d pp l q d o l o ce o e co e d e l e bo d ed e d ve ce e po ble o eep d l e e bo ow F e e bo wo ld ve o e d o o e e d 5 l e o e wo ld eep co e old l e o o e l q d I c o e w ed o b ce e l o o l q d be wee d e o e ow e wo ld p oceed ollow

e o e cle wo ld d p e bo o e o e co e ll o l o ce

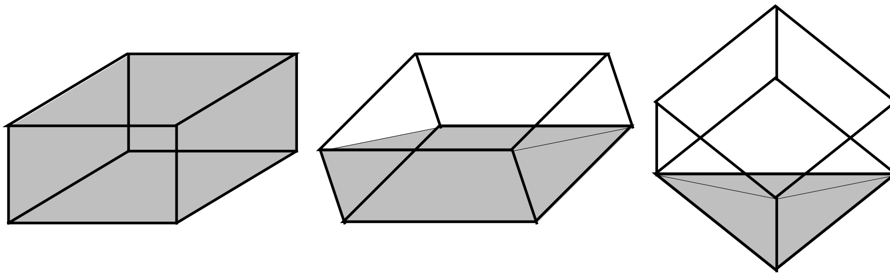
e wo ld e e p ce o o l q d l e el o e o e co e d o e c o e

Fo e ple o e e 5 l e e cle wo ld ll e e bo o e d e po o e c o e co e l l e o l q d w

* lly s pp by C CyT– p j s

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 31-40, 2001.

le e e bo e e l q d wo ld o b c o e o e co e o e e l e e wo ld ll e bo e e wo ld po l q d o e c o e co e l l e we e le e bo e e wo ld po wo l e o e o e co e d e e l e o e c o e co e



i . . s n 6 n l s

I p pe we e e e ed d e dev ce w o d o w c eve ele c be ed o e e l o o l q d p o e ll c p c b e p oced e de c bed bove e c ll e e *i r* *ri g d i* e de e e e l e vol e o ve l e dev ce w *ri g r* b e Mo e p ec el we de e e e d e o o ve l e dev ce o vol e ob ed b c l c l de pe pe d c l o e − pl e b e − pl e d o e pl e c e c l de bove b e I e c e o ec l c l de we co de dev ce l o o e ow F e 5 e w ll e e le o e ed e o e dev ce w c e co ed e ed e o e o l l o ec l c l de e 2 3 o 2 3 e pec vel e e w ll be c lled e *h igh* o e dev ce e ow e l e po ble vol e o dev ce w l b e e ve l o ob ed dev ce w ec l b e e e p o 9 l e e e o e b e o e e dev ce e d w le e e e d 9 o 9 e pec vel

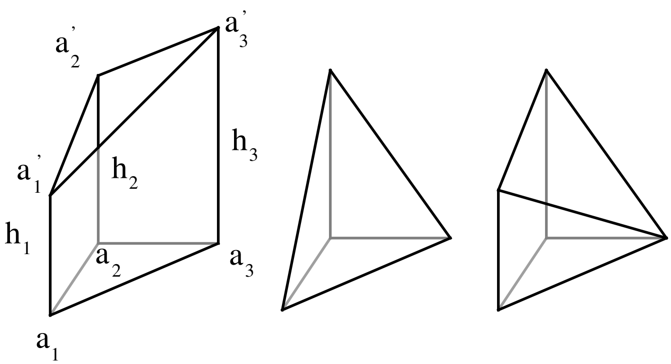
i e e i e i e wit t i e

I ec o d e e we p ove e vol e o ve l e dev ce w l b e o e e p o e e o e b e c be e oved e l e p o w ll oweve pl o l e e de w ll oo ee U de e c o e e o ve l e dev ce w l b e e d

. n nt t n t n p

e be b de e e vol e c be e ed co e po o e dev ce e p ove e ollow eo e

. *d i i h r , d h igh* 2 3 *h h i g i id r d* 2 3 + 2 + 3 2 + 3 + 2 + 3



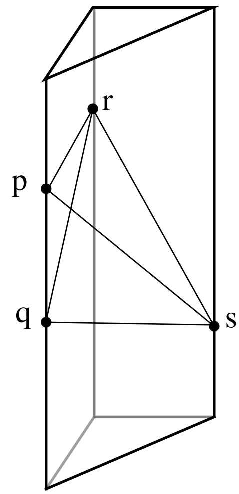
i . . T l s p ly n s n n sp ly

e e ve ce o e l belled 2 3 2 3 ow F e e e d ce be wee *i* d *i i* I well ow e vol e o e ed o w b e e d e $\frac{A}{3}$ ce e e o e b e o e dev ce e e e o e le w ve ce 2 3 ollow ed el e vol e o e e ed o w ve ce 2 3 *i i*

e ow ow e vol e o e pol ed o w ve ce 2 3 *i* d *j* p ec el *i* + *j* / e e e ollow ob e v o

e be l c l de d e e e o e le ob ed b c lo pl e pe pe d c l o ed e e we c oo e wo po *p* o o e o ed e d wo po *r* d o e e c o e e ed e o e e vol e o e e ed o w ve ce *p r* eq l o e d ce be wee *p* d ee F e

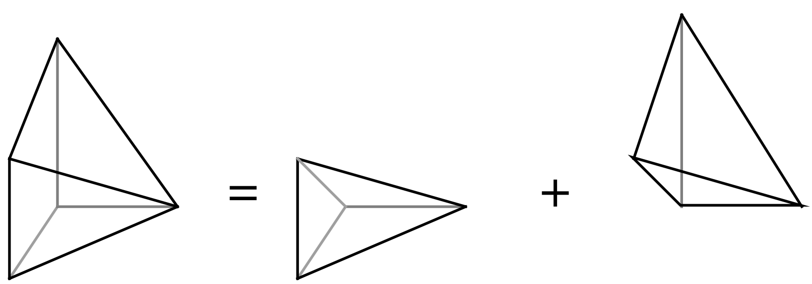
ppo e w l o D ec e pol ed o w ve ce 2 3 2 o wo e ed w ve ce 2 3 d 2 3 2 e pec vel ee F e e vol e o e e ed o d b o



i . 3. n n fix l s s r n ns ns n l ss p s ns r n

p ev o ob e v o e e o e eco d o e e d ce o $_2$ o $_2$ w c $_2$

U l e we c ow ow e vol e o e e dev ce + $_2$ + $_3$ d e e l p oved ⊔



i . . ss n p ly n s n l s n sp ly

e e o e tio

U e p ev o e l ollow $_2$ d $_3$ we c e e e ollow o o ed c e o de

2 2 3 3 + 2 5 + 3 2 + 3
9 d + 2 + 3
ow co de b eq e ce i i_k o e eq e ce

5 9

e c oc e o b eq e ce o l e c be e ed
w co e w e d ollow

i_k i_k + i_k i_k + + i o k odd

o

i_k i_k + i_k i_k + + i i o k eve

Fo e ple 9 9 + I 5
e 5 + 7 F o o w d o develop
e od o e e l e e dev ce wo e ple w ll ce o
ll e
I 5 e o e e 7 l e we p oceed ollow
F ll e e dev ce d po b c l e o e o e
co e ow e p 5 l e o e c o e co e l 5
l e e le e e dev ce e po 9 5 l e o e
o e co e e po l e o e c o e co e d
ll e p e e l e o e o e co e

I 9 o e e l e ll e dev ce e e p
9 l e o e c o e co e e po 9
l e o e o e co e d ll po e e l e o e
c o e co e e e de ow e l ve o e e
e e b e o 5 9 c I
e e ec o we develop e e o ve c

Ob e ve ce e e e e c l b e o 2 3 + 2 +
3 2 + 3 + 2 + 3 ollow e be o l e
ve l e dev ce w l b e c e e o
e e p e co e po d o l e
I e e ec o we ow e l e vol e ve l e
dev ce w l b e c ve p ov e e
e o ve l e dev ce w l vol e d l
b e o e

. x t

e ow p ove e vol e o ve l e dev ce w
l b e o e e e e o l wo po ble o de o

e e ble q e o c dev ce e e e

$$2 \quad 3 \quad + \, 2 \quad + \, 3 \quad 2 + \, 3 \quad + \, 2 + \, 3$$

d

$$2 \quad + \, 2 \quad 3 \quad + \, 3 \quad 2 + \, 3 \quad + \, 2 + \, 3$$

Co de e eco d o de d e e eq e ce o ed b e d e e ce o co ec ve ele e o o de 2 2 3 3 + 2 2 d

Ob e ve o b eq e ce o

$$2 \quad + \, 2 \quad 3 \quad + \, 3 \quad 2 + \, 3 \quad + \, 2 + \, 3$$

we c e p e o o e b e o e

$$+ \, 2 \quad + \, 3 \quad + \, 2 + \, 3$$

e

$$+ \, 2 + \, 3 \quad + \, 3 \quad + \quad + \, 2$$

w c eq l

$$+ \quad + \, 3 + \, 2$$

I e o ee o b e o i 7 we c oc e q e b eq e ce o

$$2 \quad + \, 2 \quad 3 \quad + \, 3 \quad 2 + \, 3 \quad + \, 2 + \, 3$$

e be o e e c be o ed w b eq e ce o

$$2 \quad + \, 2 \quad 3 \quad + \, 3 \quad 2 + \, 3 \quad + \, 2 + \, 3$$

eq l e be o ele e c be ob ed e o e ele e o b e o i 7

De o e b 2 d 3 + 2 o e be ob ed e o e ele e o b e o i 7 be o e o + + k w e e d k bec e 2 d 3 + 2 epe e elve e e o d e e ce d e e pec vel I ollow ow e c p c o dev ce c e o 5 × × 9

ow co de e eq e ce

$$2 \quad 3 \quad 2 \quad + \, 2 \quad 3 \quad 3 \quad 2 \quad 2$$

o d e e ce o co ec ve ele e o

e p ev o p p le 2 3 2 d d + 2 3 I c e be e e ed b o e o + + $k + d$ w e e k d bec e

2 3 2 d + 2 3 epe e elve e eq e ce o d e e ce d e e pec vel ppe bo d o e be o d c be e e ed w e eq e ce l e × × × 5 e ow p ove o e e q e e epe ed o e d o e d e

$$+ \quad + k + d \quad + \quad + \quad + \quad k \quad + d$$

w e eve k d e e o e e be o epe o d c e o 5 be c be e e ed O e l ow ollow ce e dev ce w l b e o e d e d we c e e o o l e

e e ti eq e e

e be eq e ce o be c i i k Fo e c b eq e ce i i o we oc e e be i i + i i + + i odd o i i + i i + + i i eve Fo e ple 9 7 5 d 9 e + 9 C ll g r i g o e c e e e e b eq e ce o c e be ow e e eq e ce e e o e co e po d o powe o e I ec o we develop l e e e o dec de eq e ce o be e e eq e ce

e d le be e eq e ce ob ed b o e eq e ce i i k Fo e ple e eq e ce

9 7 5

e

5 57

e ow p ove

. h i g r i g i d i i + + i , k

U e p ev o e l e o ve

9 7 5

e e eq e ce ce

+
+ +
+ + +
+ + + +
+ + + + +
5 + + + + + +
57 + + + + + + + 5
+ + + + + + 5 + 57

e ow p oceed o p ove e e l e Ob e ve
i j i i + i i 2 + + j j o b eq e ce
i i o c be w e o ele e o Fo ce
e p ev o e ple

9

+ 9 +

w c eq l

5 + 5 + + 9 + + +

e

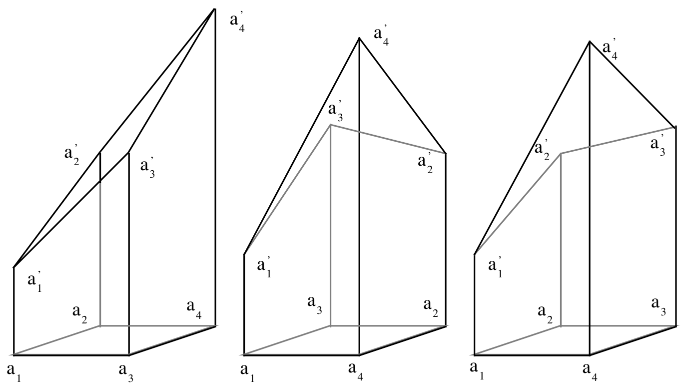
57 + + + 5 + + +

pl e e e eq e ce e e o o c be
w e e o o e ele e o
ppo e e i + + i k e p ove
e e eq e ce o e B d c o e e e
be wee d + + i c be e p e ed e o o e b e o
i Ob e ve e e o e o m+ i m + + i
c be e p e ed e o e ele e o b e o i i ce
b po e i + + i ollow e e le o
eq l o + + i c be e p e ed e o e ele e o o e b e
o i
Co ve el o o e we ve i + + i e cle l
o e e eq e ce ce + + i + c o be e e ed e
e l ollow

5 De i e wit e t e

I ec o we d ve l e dev ce c e e p o 9
I q e b e o e d e d 9 e ve b e
de c p o o ow w ob ed ce e de ed e ve l o o e
ed p ev o l we w ll o l e c e e

e c e o l b e dev ce we d o e o e o o
l q d c be e ed d ec l w dev ce o e $_2$ $_3$
dd o l p oble e ce e e e be o d e e pe o e
dev ce e l d e e e o e ble q e e c e o
l b e dev ce we e e ve ce o e e dev ce e l beled
$_i$ $_i$ o w e e e d ce be wee $_i$ d $_i$ $_i$ ee F e 5
Co de dev ce o e $_2$ $_3$



i . . p s s n s s b s

I we be l e e o dev ce w e cloc w e
o de e e e e o d e e pe e dev ce c ve el
$_3$ $_2$ $_2$ $_3$ $_3$ $_2$ $_3$ $_2$ $_2$ $_3$ d
$_2$ $_3$ o ce d eco d d o d d d
dev ce ve e e e o e ble q e p o e e e
e o l pe e eleve o e e ow F e 5 I e b
ed o wo o c ec e dev ce w e $_3$ $_2$ c e e
e ollow o $_2$ $_2$ + $_2$ $_3$ $_3$ + $_3$
d I dd o + $_2$ + $_3$ we c l o e e $_2$ + $_3$ Fo
e ple c be e ed b e po $_2$ $_3$ d b $_2$ d
b e pl e p llel o e b e p o d o o Fo o e
o e dev ce ow F e 5 e e e o e vol e c be e ed
be e l d co ld be p od c ve o c e e e c p c o
ec l b e ve l e dev ce F e wo lo e e l e
p o e e c ze ll e bove ob e v o e e eo e

. *h r i i r ri g d i r g r i h*
r 6 h + $_2$ + $_3$ *h r h i g*
, , , $_2$, $_2$, + $_2$, $_3$, $_3$, + $_3$, $_2$ + $_3$,

v e e e l d we we e ble o d dev ce o o c p c
9 b b e o ce e d d b w C p o d d e ollow
o ve e o e 2 3 co p e e
2 2 + 2 3 3 + 3 2 + 3 o e ble
q e e e ppl ed e c e o o eo e o e b w c we
o d wo ve l e dev ce w e 9 d
9 c e e ll e e q e o o 9 Fo o e e o
e e e o e ble o ob ed we e o e e o d
lle c p c e
we ve

. h r i i r ri g d i i h r g r hi h r 6 i r

n wle emen : l l n n b s n s
n n n p p

Re e e e

y n s n n
b : *M d s d x s M s H g S d H g S S d s V* n pp n n p n s))
: *W M v s d?* n b n b
l s s n p n s))

h l s ss
S ll n s l S

J o oz [2] d e c Oz w [3]

s ns n l l p n T n s y
T y b y T y 5 006 p n
w m n we ne
l l n n s s n s n s y
T s T y 06 5 0 p n
n ak ma a
s n s l
n s 5 55 p n
awa a -ne ne

a . ss n s n lly n s bl s s p n s n n b p ly ns n n ss ly s s n b n s s s s n s n p n s s s ss ly ss n s ll k k p s n n s x n b n p s s s by n n n l ss n yp l y n n p l "p ly s ss n yp s n x p ss n yp n s n s ll s s n yp p ly s ss n s n lly n s bl s T y n s ss n s p l sp yp n p ly s ss ns

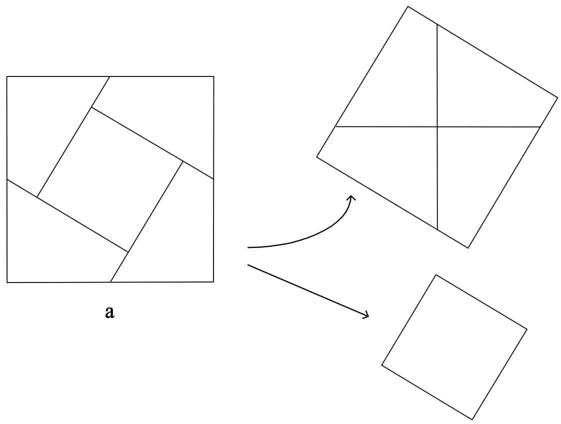
t o tio

I e e p zzle o deco po e q e o be o q e F ow ple w o deco po e q e o wo b q e d F ow d ec o o q e o e p ece w c c be e ed o wo q e ee q e d o q e e c ll d ec o o eq e ll d v ble q e B c op o d d ec o o eq e ll d v ble q e D jve pe o d q e c be d v ded o p ece ll o w c e q e o d e e ze

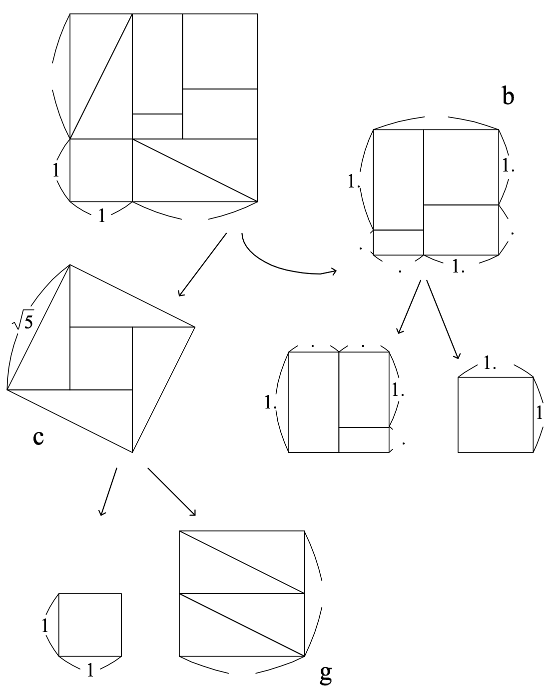
Oz w 5 o d e e w o d ec o ow F w c c be ppl ed epe edl eve l e e e l d ec o o eq e ll d v ble q e we c e e e p ece o wo q e ee q e d o o p o q e cce vel

e be o p ece e d ec o ve B e eco d d ec o e b q e o be d ec ed co o e c l e l e d ee F d e e o e e o l be o p ece c e e o l b ee eve

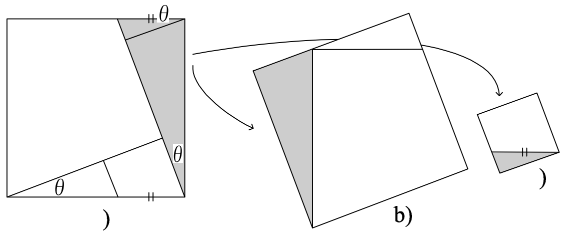
J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 41-52, 2001.



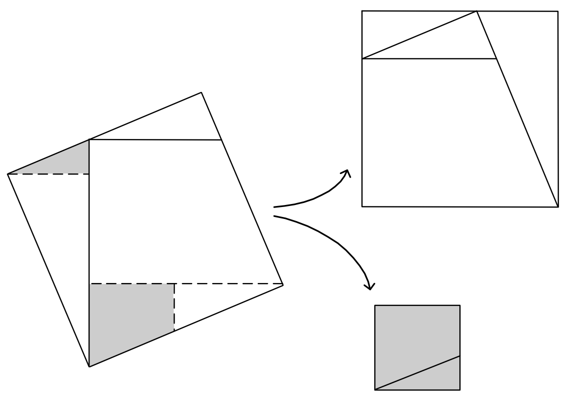
i . . s) s n fi p s s s b) n)
ns



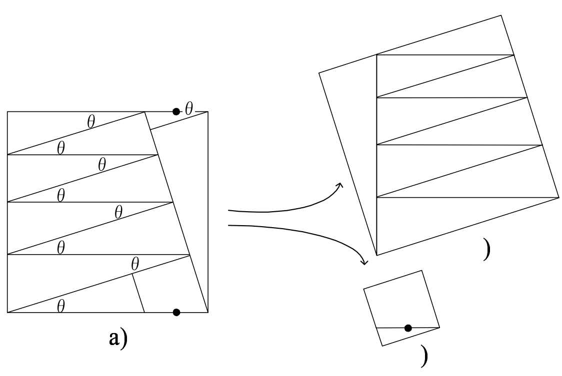
i . . s s n n n p s n ns s s
b) n) s s)) n) n s s))) n)



i .3. T sy b l s n s n b y n l b n 0° n 0°



i . . T s b) n F s ss by pply n s “ ss n p n
) n F p p n lly n n n l n s s l n
s n s b n ly ss



i . . T sy b l s n s l b ly s n n l °)

epe o o e d ec o p ov ded e le bee c o e c e ll depe d o e de ed be o b q e c e co d o o o ob b q e eco o c ll ollow

Fo eve e e p o le

$$\text{co}\ ^{2p} \quad +$$

ble Oz w ow e v l e o e eq l e bove co d o o e c p e e wo B e e v l e α d e co d o c be ep e e ed o e e pl c l ollow

$$\alpha \quad \text{o} \qquad 9$$

a le . 6])

	°)	°)
6	6	
0		0
	0 6	50
		6
6		

d o d depe de l e e e ll e d ec o ow F 5 d o ced e ollow c

B α o e o l be o p ece c e e o l b wo o e c ppl c o o e d ec o w c c be ppl ed ec vel o e

e p o e b q e ve e e ze B w e p ll b q e ve d e e ze

ow le f be e be o p ece eq ed o co c p o b q e eq e ll e d ec o e od pe k —— e d o k o l e Oz w e od pe w le e od pe oz ve pe p oced e o eq e ll d v ble q e

oz p oced e e o e ee e od e d ec o F D jve pe p o d Oz w d ec o I op l w e pec o e pe b o o ple d ele Oz w e od d e od I w ollow we de e e o o o p el ec ve d ec o d p ove e od op l w e pec o e pe o e p el ec ve d ec o

i otio

fin t n . *di i r i di i i r i r i i r i r g hi h r rr g d i r , hr r , d , r i*

fin t n . *f m h r i r ir d r m r —— d k r rg m, h di i i id k*

fin t n . *di i r i g r r r i g di i g r i i r i g r h i g h r rr g i i r*

e p e F d ec o p e d e p e b c e co po o p e

fin t n . *di i r i di i i r i id r r i i i i h i g di i*

i i d r r i i i d di i r h r i i d r d r r r i d h i , i i h r i g r

h r rd , i i r g r i r d h r di d

B h r i i , r d r d h di i r i r r i i d h r ,

o e ppl o e ec vel we ve ve b q e ollow

d

B epe e e p oce e we ob d ec o o eq e ll d v ble q e

i id r r r i i h i g di i i i d

h i i r i i i r h

d ec o p el ec ve d ec o p e e p o F 5

rk e e we do eq e e ze o b q e e d e e

ow we c e o eo e

. *h r i r r r i di i r i di i i r , h i h* ⊔

. *ki k r ' di i i i i h r h , g h r r r i di i* ⊔

P oo o eo e

. p t n d t n

ppo e e e p el ec ve d ec o o eq e ll d v ble q e w o e pe le wo e be e d ec o p e d e co po o p e o

ppo e co p ece

d $_s$

e dec e o de o e ze B ppl e d ec o p e o we ve wo b q e d ce e co po o p e o l o o e be o p ece o eq l o o d e e o e e o l be o p ece c e ed b e be k o p ece I eve e ppl c o o e o l be o p ece lw c e ed b k ce e pe o e d ec o ed o be le wo k o ld be eq l o o e o e p ece q e d e d ec o pe ow le

d $_s$

be e p ece o e dec e o de o ze ce co po o p e l o o e o o e ze o $_i$ d $_j$ eq l o e o o e ze o $_i$ d $_j$ ce k o l o e o o l p ece $_i$ d v ded o wo e p ece d v ded o ld be $_j$ d v ded o o e e $_i$ $_i$ o ll d $_j$ lle $_j$ e e o e e o o e ze o d $_j$ o eq l o e o o e ze o d $_j$ B l e ow d v ded o q e d $_s$ d $_i$ $_i$ o Mo eove $_i$ e l pol o w c e ow o be ec le O co e e o o e le o e wo de eq l o e olde o τ

$$\tau^2 + \tau \qquad \tau \quad \frac{+ \quad \bar{5}}{}$$

ble c le we c e w o lo o e e l e le o e lo e d o e ed e o e d τ e pec vel e e le o e lo e ed e o eo e c l eq e ce $_j$ w e e de o e e oo o τ

ce e p ece e ec l obv o e e wo e e o e q e eq l o

$$\tau \quad + \ ^2 + \quad + \quad + \ ^{2\,s}$$

O e o e d e le o ed e o e o e le o ed e o o e $_i$ o we ve e ollow eq l

$$\tau \quad + \ ^2 + \quad + \quad + \ ^{2\,s} \qquad \sum_{i=}^{s} \ _i \ ^{2}_{\ i}$$

w e e $_i$ o τ e lo e o o e ed e o e p ece co ed
e ed e e pec vel d $_i$ o e w e
e ll ow eq l w ll r old

. n t t

. *h i g i i r h d* **Q** *τ i irr d i*

^{s} τ ⊔

ele e p oo w ll be ve e ppe d
ow ppo e e eq l old

$$\tau \quad + \quad ^2 + \quad + \quad + \quad ^{2\,s} \qquad \sum_{i=}^{s} \quad {}_i^{\;\;i} \quad {}^2$$

$$\sum^{s} = \sum_{i \; j=} \quad i \; j + \tau \quad \sum_{i \; j=s} \quad i \; j$$

w e e d $_i$ o τ

. e be odd
I c e we ve e ollow eq l

$$\tau \quad + \quad ^2 + \quad + \quad + \quad ^{2\,s}$$

$$\tau \quad + \quad ^2 + \quad + \quad + \quad ^s \quad) + \tau^2 \quad + \quad ^2 + \quad + \quad + \quad ^{s\;\;3})$$

e e o e e co e o e le d de τ O e o e d e
co e o e d de

$$^2 + \tau \sum_{i \; j=s} \; _{i \; j} \quad i \; j$$

w c o ld be b e eq l o e co τ e le d de B
c ll c be w e e o

$$\tau + d \quad + \quad \tau + d \quad \text{o} \quad \tau \; + \quad \tau + d$$

cco d o e v l e o o o e e e d d B o e o e c be
eq l o τ o e eq l c old c e

. e be eve
I c e d

$$\tau \quad + \quad ^2 + \quad + \quad + \quad ^{2\,s}$$

$$\tau \quad + \quad ^2 + \quad + \quad + \quad ^{s\;\;2}) + \tau^2 \quad + \quad ^2 + \quad + \quad + \quad ^{s\;\;2})$$

$$\tau + \tau^2) \quad + \quad ^2 + \quad + \quad + \quad ^{s\;\;2})$$

$$+ \quad ^2 + \quad + \quad + \quad ^{s\;\;2}$$

o b e e ollow le ed e

. *h h i h d , d i*

$$i\ j + \tau \quad i\ j \quad r \quad k, \quad d$$
$$i\ j= \quad i\ j=s$$

$$2 \quad i\ j + \tau \quad i\ j \quad r \ dd\ k$$
$$i\ j= \quad i\ j=s$$
⊔

. *h h di i 2 i 2 i id h*

$$i\ j$$

$i + i\ dd$

r e de o e e be o p c l e $i\ j$ e
e + e ollow e

p e be o e $i\ j$ c $i\ j$ τ^2
e be o e $i\ j$ c $i\ j$ τ
r e be o e $i\ j$ c $i\ j$

e de o e b p d r e be o co e po d e e eco d
e + e

$$i\ j + \tau \quad i\ j \quad p\tau^2 + \tau + r + \tau\ p\ \tau^2 + \tau + r$$
$$p + + p + r \quad \tau + p + r \quad p +$$

e e o e

$$p + + p + r \quad d \quad p + r \quad p +$$
$$p + + p + r \quad p + + p + r + \quad + r$$
$$p + + r +$$

ce ll e e v l e p e c e o e ve

$$p \quad r$$

I ollow ed el p r o $i\ j$ o + k o + $+ k$
w e k odd
ow + odd e bo k + d k + e odd ce
eve d e e o e $i\ j$ ⊔

. i / r , h j r r dd ⊔

e de o e

$$i\ j + \tau \quad i\ j$$
$$i\ j= \quad i\ j=s$$

e we ve

$$_{2j} \quad {}^2_j + \tau \ {}^2_- \ _j + \quad {}_{i \ j=} \ {}_{i \ j} \ \ {}_i \ {}_j + \tau \quad {}_{i \ j=s} \ {}_{i \ j} \ \ {}_i \ {}_j$$

. h $_{2j}$

$_j$, h $_- \ _j$ τ

2 $_- \ _j$, h $_j$

r e e p e o

$${}^2_j + \tau \ {}^2_- \ _j$$

eq l o

$${}_i \ {}_j + \tau \quad {}_i \ {}_j$$

w c c be ep e e ed e o $\tau +$ o o e e e d
I $_j$ e e v l e o e bove e p e o eq l o τ o
τ cco d o e c e $_- \ _j$ o τ e pec vel o o l e c e
w e $_- \ _j$ τ po ble
e p ope ve ed b l e ⊔

. . $_i$ / o o e eve
I c e $_j$ o eve odd e e o e b e $_- \ _j$ τ
o odd e $\frac{s}{2}$ odd o e w e $_- \ _j$ $_- \ _j$ o eve
e e d $_j$ o eve e e le $\frac{s}{2}$
e ll $_j$ o eve e e le $\frac{s}{2}$ d $_j$ τ o o e eve
e e o e

$$+ \ + \ \tau \quad {}_{i \ j=s} \ {}_{i \ j} \quad {}_i \ {}_i \quad + \ \tau \times \frac{\frac{s}{2}}{\quad} \times \quad \times \tau \quad + \ - \quad \tau^2 \ /$$

ce co d c e p ope e

. . $_i$ o eve eve e e
B e p ope e $\frac{s}{2} +$ τ o eve eve d $\frac{s}{2}$ odd
o e w e $_- \ _i$ c e e $_- \ _i$ o odd d $_j$
o eve odd le $\frac{s}{2}$ B o we ve

$$_2 \quad {}^2 + \tau \ {}^2_- \quad + \quad {}_2 + \tau \quad {}_{i \ j=s} \ {}_{i \ j} \ \ {}_i \ {}_j$$

$$+ \ \tau \times \frac{\frac{s}{2}}{\quad} \times \quad \times \tau \ /$$

O ce o e e p ope e v ol ed

we c co cl de e eq l c old c e

kn d nt e o e ve e l o P o e o F ede c o o v e v l ble o o el o d d o P o e o o o U ve o elp co ple p pe o

Re e e e

y : n n ss n s n lly *n* s bl s s n C p n l y s p T n s y) 0–
ss p : bl s ll C sp n n 6) –
: T l bl n p n s) n " n n l b l s n y n sy) – 6
: n ss n s n s s n p n s) n) 5 –56
5 : n n n l ss n p n s) n 0) p
6 : p n n)

pp nd x

h i g i r h d **Q** τ *i irr d i*

$$s \quad \tau$$

h r d $\tau^2 + \tau$

F we ll ow e ollow c

t . *m i g r gr r h h , h m h r* τ *i i* **Q** τ

r ppo e **Q** τ

$$p \tau +$$

o o e *p* d **Q** Fo b e e we p

$$\tau + \qquad \tau +$$

Obv o l bo d e pol o l w e e coe c e Fo ce

$$d$$

ce

$$\tau + \; ^2 \quad \tau^2 + \quad \tau + \; ^2 \qquad \tau \; + \quad \tau + \; ^2 \qquad \tau + \; ^2 +$$

we ve

$$ {}_2 \quad d \quad {}_2 \quad {}^2 + $$

I e e l

$$ \tau + \quad \tau + \quad \tau + \quad + \quad + \quad \tau + \quad + $$

d e e o e

$$ + \quad + \quad d \quad + $$

ow ce o c pol o l w e e coe c e o l ol o c o e eq o o ld be e e o we ll ow / o e e d

.

e $\frac{}{A}$ I c e $\underline{c}$ Be de e

$$ \frac{+}{+ \quad +} \qquad \frac{+}{+ \quad +} \qquad + \frac{+}{+ \quad +} $$

o o c e

.

I c e $\underline{c}$ I e

$$ \frac{+}{+ \quad +} $$

ce e e o po ve d e de o o e ve o c e

rk I e b d / o

e ll / d e e o e / o o *p* $\tau +$ / τ l o w de ed o be e *m* oo o τ Co d c o ⊔

t . *i g r gr r h , d h h r* τ *i i i g r h h* ^{t} *i i* **Q** τ

r e *d* be e e e co o d v o o d I ^{t} **Q** τ e o ce ^{s} τ **Q** τ B po ble ce e $\underline{s}$ oo o τ ⊔

ow e pol o l c o zed ollow

$$ s \quad j) \quad j= $$

w e e p ve oo o I ed c ble $\mathbf{Q}\ \tau$ e d v ble b o c pol o l R ove e eld $\mathbf{Q}\ \tau$ w c c be w e ollow

$$ R \quad j) \quad j $$

I co e $\mathbf{Q}$ e p od c o $j)$ d w e e ollow o

$$ t \quad t \quad j $$

w e e e de ee o R e e be o ele e ce e b ol e v l e o d ppo ed o be $\mathbf{Q}$ e p od c $\prod\ j)$ eq l o o Co eq e l t $\mathbf{Q}\ \tau$ e p ope ed F c e pol o l ed c ble $\mathbf{Q}\ \tau$

S ll s l ss s S l l s

J o o [2] d Jo e U [3]

s ns n l l p n T n s y
T y b y T y 5 006 p n
w m n we ne
s ns n T n s y
T y b y T y 5 006 p n
aka - ka a
ns s C n s
n s n l n x x F x
rr a ma nam mx

a . k ss n $\mathcal{D}$ p ly n s p n n s s bp ly ns sj n n s s s n b ss bl k p ly ns ll s l y k p s $\mathcal{D}$ n b ss bl n p ly ns ll s l n $\mathcal{D}$ s ll s n lly k s bl ss n n s p p s ny n x n n n 5 s s n lly k s bl ss n k)n p s s n lly k s bl ss ns s l p ly ns n 6 s F s ny s pl p ly n n s s k) ss n n) k n −) p s k 0 n b ss bl k p ly ns s l s l s l s s s p p ly ns

t o tio

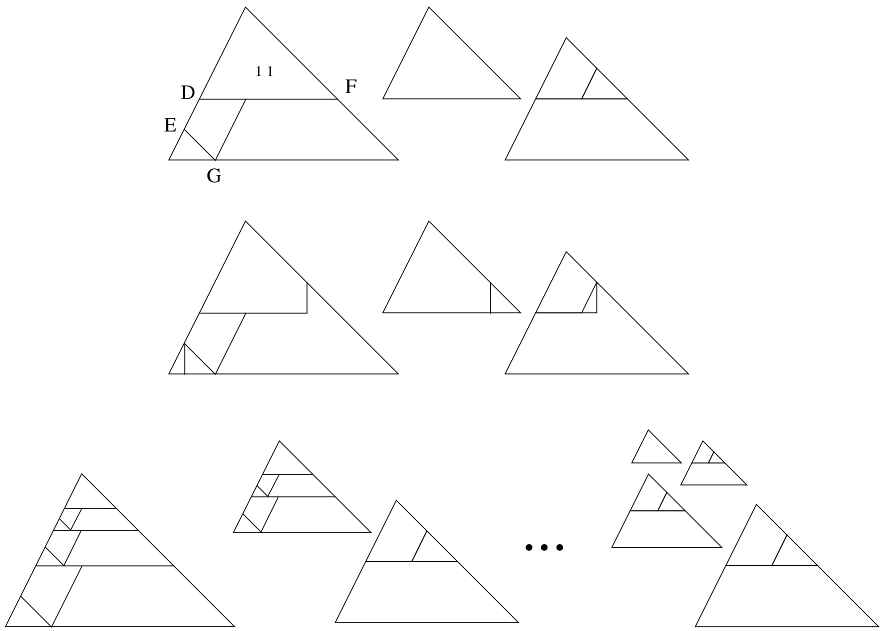
D ec o o pol o l cl c l eld o d e e c l c e ce cl c l e l o e 9 ce b ow ll ce Bol d e w e e ve wo ple pol o d $\mathcal{Q}$ o e e e we c d ec o e be o pol o w c c be e e bled o o $\mathcal{Q}$

Boo e o d ec o o pol o ppe o e o e e l e e e c b ew dv ce d e e p zzle o e op c we d e e e Fo e 9 7 c 9 9 d e 9 Boo b d e 9 d F ede c o 7 997 devo ed olel o e d o d ec o

e be pol o o e pl e *di* *i* o p o o o m bpol o c i j m w e e de o e e e o o c i c lled *i* o I o e o e p ece o l o w ll be c lled *ri i* d ec o o

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 53-66, 2001.

k di *i* o p o o o bpol o
w d jo e o c e c be e e bled o o *k* pol o ll
l o d ec o o c lled *i* *k di i i* o eve
k p ece c be e bled o o o pol o l o I
F we ow d ec o o le F b d c ow o v l
d ec o d eq e ll d v ble d ec o o e e le



i . . ss n n n l ss n n s n lly s bl ss n
n l

Coll o 5 ee l o p e 97 o 7 ve d ec o o e l o w
+ o p ece o odd o eve e pec vel e e d ec o llow
o co c eq e ll *k* d v ble d ec o w *k* + o *k*
k + p ece e pec vel eo e p pe eq e ll *k* d v ble
d ec o o q e ve bee d ed − I p pe we p e e
eq e ll *k* d v ble d ec o o le co ve q d l e l d co ve
pe o w *k* *k* d 5*k* p ece e pec vel Fo le we
p e e o v l eq e ll *k* d v ble d ec o w *k* p ece Fo
e l pol o w *m* ve ce we p e e d ec o w p ece
w c llow o co c eq e ll *k* d v ble d ec o w *k* *k*+

p ece Fo *i* *g* o ece l co ve w ve ce we p e e
d ec o w p ece llow o co c $k+$ d ec o
w o $+k$ $+\frac{}{3}$ p ece F ll o *r h d g*
we ow l e l

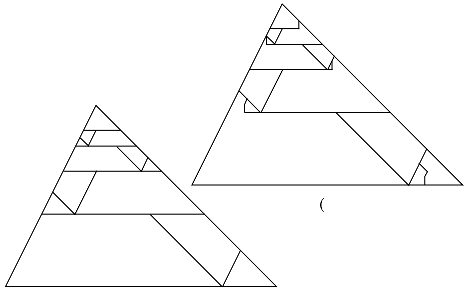
e ollow o o w ll be e l o o e p pe ve wo po
d o e pl e w ll de o e e l e e e jo e e po
$+$ w ll be de o ed o e d e e o
Fo e ple w e we ob w e we e d w e
$\frac{}{2}$ we ob e d po o e e e l l le Q be pol o l
e p pol o w ve ce e Q w ll de o e e pol o l
e p e pol o w ve ce I l e e e d
R e p llel we w ll w e $\|$ R I pol o ve ce
we w ll o e e e o e pol o w ve ce o pl e
pol o
wo pol o d Q e c lled *gr* e e l o
o o R d pe p e ec o p o o Q d Q e c lled
i i r e e pp *f* ^{2} ^{2} c *f* *p* $+$ w e e *p*
po ^{2} d c *f* d Q e co e Ob e ve wo
pol o d Q e l d ec o o d ce
l w d ec o o Q c e p ece o e e e f_i
m D ec o w ll be e e ed o e *di i i d d* Q b

eq e ti i i i e i e tio o t i e

o le we ow e ollow eo e

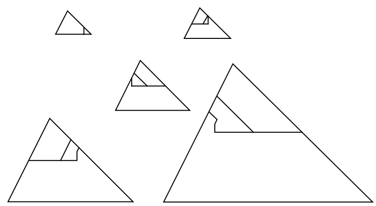
. *ri g h i k di i i di i r*
ri i i k di i i di i i h k i r k

r e be le w ve ce e be e d ec o o
ob ed ollow e d be e po o B c $\frac{3}{}$
d – d le d be e po c
$\|$ $\|$ d $\|$ F e le $_2$ w
ve ce l o d e o o l de $\frac{3}{}$ o e
le $_2$ $_2$ w ve ce d pezo d $_2$ $_3$ d $_2$ w ve ce
d e pec vel c be e bled o le
l o w o o l de eq l o – ee F
I ec ve w le $_j$ be e d ec o d ced $_j$ b w e e
$_j$ e le o $_j$ co ve e Fo ed e e
k ^{2} ^{2} ^{3} ^{2} de e
d ec o o w e c l k $+$ k p ece ee F c
Cle l eq e ll k d v ble d ec o o
F d o v l eq e ll k d v ble d ec o o le o e
c lle F we b od o ob eq e ll k d v ble



((

i . . F n n n n l s n lly 5 ss n n l

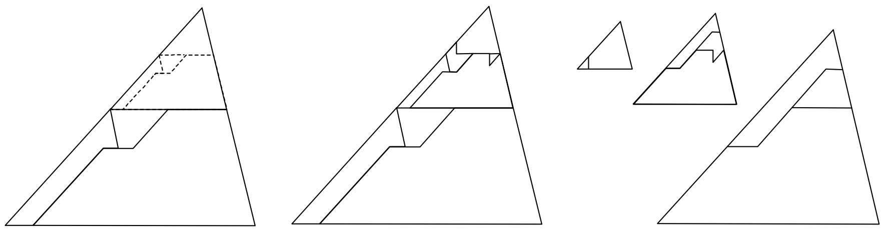


i . 3. ss bl n ss n n F b) n fi n l s

d ec o o le ow F e objec ve o
o od c o o e e eve eco d le o o op o
bo o o c e d e o e o c w e e cep o e le
co w c o c e d e de l o od c o e
o w d d e le o e e de e ow p oceed o ow ow we c
od co c o ob o v l eq e ll *k* d v ble d ec o
o o le

ppo e we el bel e le o o op o bo o b
pl $_2$ o wo le $_2$ d $_2$ b d w e e o op ve
e c e l le e o l o le Jo le
$_2$ e le p ece o $_2$ o e ele e o below o o p
p le e p p ece e p ece v e p o v ed e
co ed w e pl o wo p ece o e o w c e
p ece l o $_2$ d jo e p ece o e ele e o below
ow F b I ec ve w we ow pl $_i$ o p ece $_i$ d
le p ece $_i$ c odd e $_i$ co e w $_i$ d
eve e $_i$ co e o $_i$ *k* e odd jo $_i$ o e
p ece o below el e eve jo $_i$ o e p ece o below *k*
ee F b I ow e o ee e p ece o e d ec o ob ed
o eq e ll *k* d v ble d ec o I F we ow ow o e ble e
p ece o e d ec o F b o ve le ⊔

rk e c od d ec o o le d e o Coll o ee F e 9
7 d ow F ol d l e o ob d e e *k* d v ble d ec o
l o w *k* p ece ee F b



a

i . . F n n s n lly s bl ss n n l

Q i te

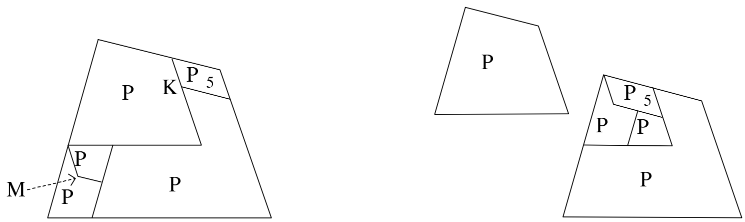
e ow ow

. *dri r h i k di i i di i*
i h k i

r e be e co ve q d l e l o be d ec ed e
e _ + _ d _ + _
e ve eq e ll d v ble d ec o o co o ve
p ece e be e po c $\frac{}{3}$ $\frac{}{3}$
d le be e po o e d o l c || o ||
e be e po c − − le
be e po c || d || d le
be e d po o e be e e ec o po o e l e p
o d p llel o d e l e p o d p llel o
ce _ + _ d _ + _ po e p llelo
p od ce e d ec o o w p ece
ow F 5 I e o ee eq e ll d v ble d ec o
o F 5 b
e w ll ow e e d ec o d ced b o ob d ec o
o B e p oce we e eq e ce o eq e ll *k* d v ble
d ec o o w *k* p ece ⊔

Pe t o

I ec o we p ove e ollow e l



i . . ss n n x l l

. g h i k di i i di i i h
5k i ⊔

e o e Coll o d ec o o e l pol o w p ece
ee F e 9 7 7 we c ob o e l o e l pe o e ow
p e e wo d ec o o o e l co ve pe o w ll p ove o
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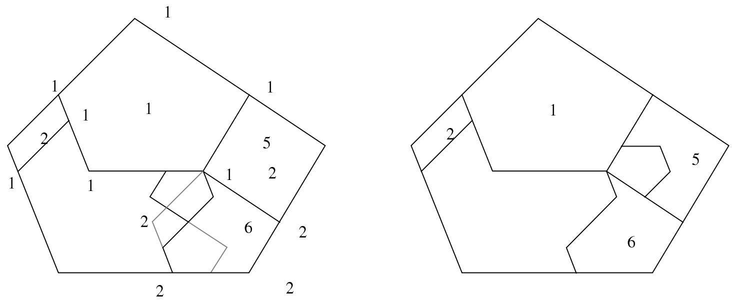

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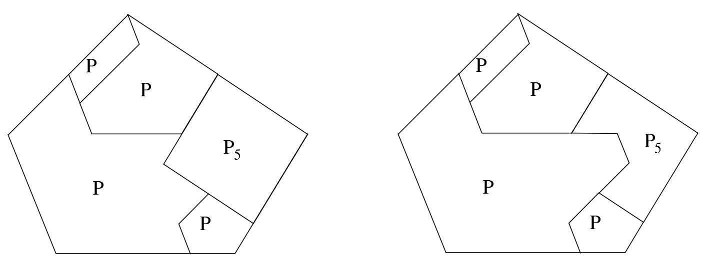

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I c e we ow belo o e e o o $_2$ $_3$ o p ove e d ec o ow F e l z ble ce

$$- \quad \boldsymbol{b} \quad \overline{3} \quad + \frac{\alpha}{2} \quad {}_2$$

d ce $\overline{3}$ d $\frac{\alpha}{2}$ e o po o e p llelo $_2$ ce $\_$ e de ed co cl o ow ollow

. $\overline{3}$ α

e w ll ow c e e d ec o ow F b e l z ble o ee we ow d e e o po e p llelo $_2$ o e α $\frac{2}{3}$ d c e

e d δ be q e e l be c $+\delta$ $\boldsymbol{b}$ $+$ δ $\boldsymbol{b}$ ce $\_$ $+$ $\_$ d $\_$ $+$ $\_$ δ d

O e o e d we ve

$$- \quad \boldsymbol{b} + \delta \quad \boldsymbol{b} \quad \overline{3\alpha} \quad + \frac{\gamma \quad \alpha}{2\alpha} \quad {}_2 \quad \text{d}$$

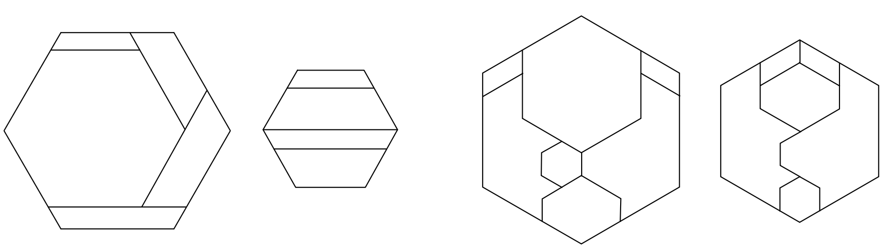
$$- \quad \boldsymbol{b} \quad \overline{3\alpha} \quad + \frac{\alpha}{2\alpha} \quad {}_2$$

e ce $\overline{3\alpha}$ $\overline{3\alpha}$ $\overline{2}$ $\frac{\gamma \quad \alpha}{2\alpha}$ d $\frac{\alpha}{2\alpha}$ $\overline{2}$ e de ed co cl o ollow

I ec ve w we e eq e ce o eq e ll k d v ble d ec o o w $5k$ p ece

5 e o

J c e l o d d ec o o e e l e o w ve p ece ee F ee l o F e 9 7 B ec vel c e l d ec o ollow e e e eq e ll k d v ble d ec o w k p ece U e d ec o ow F b we c l o ob eq e ll k d v ble d ec o w k 5 p ece



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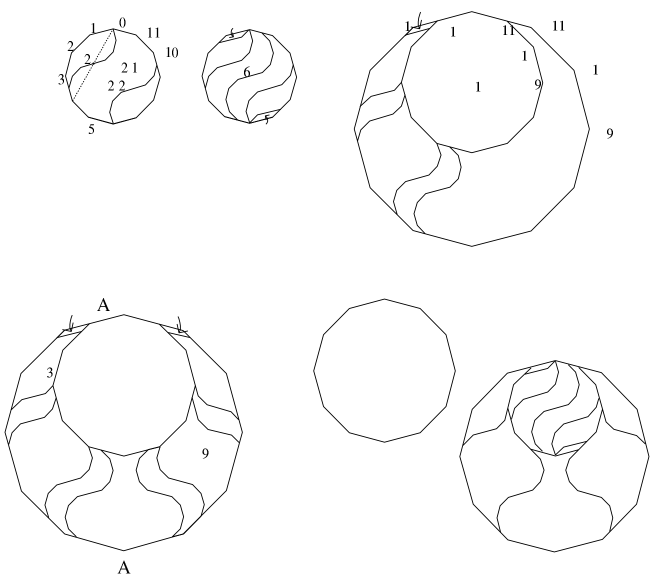
Fo e l pol o w *m* ve ce we ow ve ew d ec o w p ece w c llow o c c eq e ll *k* d v ble d ec o w *k* *k* + p ece e be e l pol o w *m* ve ce l beled e co e cloc w e d ec o w be e op o ve e o



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Co de eco d e l *m* pol o w ve ce o ze $\frac{2}{}$ o e p e e d ec o o w *m* p ece ob ed ollow

Fo e c *m* le $\mathcal{Q}_i$ be e pol o l w ve ce $_{2i}$ d le $\mathcal{Q}_i$ be e pol o l $\frac{}{2}$ $_{2i}$ $\mathcal{Q}_i$ e $\mathcal{Q}_i$ be e pol o l ob ed b jo cop o Q_i w e pol o l ob ed b o Q_i de ee o d e po $\frac{}{2}$ $2i$ ow F 9 o e c e m F ll le $Q_{i\,2}$ be e pol o l ob ed o Q_i b o de ee o d e ce e o e e o Q_i $Q_{i\,2}$ m d ce p o o o m p ece e l bel e e o o p o $2i$ m ow F 9 b l bel $2i$ m w ll co ve e $2i$ o

e ow ow d ec o o w ll co p ece l o $2i$ m e be e pol o $\frac{3}{}$ e l bel e ve ce o b i m w e e

Fo e c m le l e cop o $2i$ o ve e $2i$ pped o ve e $2i$ o ce e le o e e $2i$ $2i$ $\frac{2}{}$ e le o $2i$ e po o $2i$ p o ve e $2i$ o ee F 9 c e ow p o c e co c o lo e l e p o d 2 o ob d ec o o w m p ece ow F 9 d bel e e o $2i$ de o pp lo e l e de e ed b d 2 b $2i$ 2 m ow e e e Cle l w e we e ec $2i$ 2 m e e l p ece o e e w $2i$ m c be e e bled o o I ow e o ve e e p ece 3 c be e bled o o pol o l o o ze − e ze o pol o co e o I ow ollow e e l p o o d ec o o ee F 9 e

Di e ti i e o o

Co de ple pol o w ve ce e ow p e e d ec o o e e c l p ece l o o p o o o le 2 w d jo e o ob ed b c lo d o l jo p o ve ce o ee F e ob e ve ow we d ec e c i o l le i $i\,2$ $i\,3$ i b c lo e l e e e jo e d po o ed e we ob d ec o o w le ee F b Cle l o e c e e o le j $2\,j$ c be e e bled o ob pol o l o

e ow ow ow o od o ob d ec o o w p ece F we colo e ve ce o w colo d c w wo ve ce o e dj ce e e co ec ed b d o l o o ed e o e ece ve d e e colo ee F O ew d ec o ow ob ed o b el e c de lo e d o l ed o ob ee F b e p ece o e pol o e c o w c co e c l o e ve e o pl e o le o e o e c i ce co le ow ollow e be o p ece o e c l

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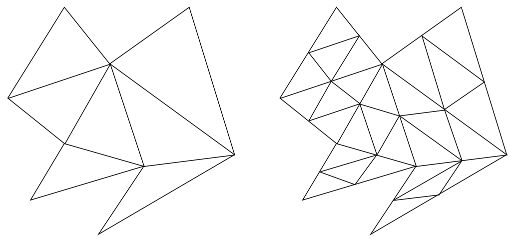

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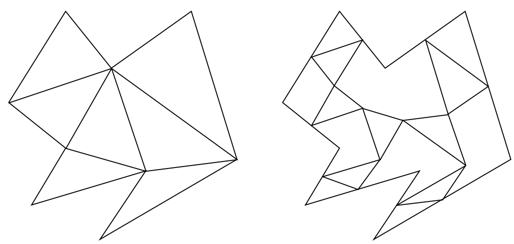

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o ce e c le e c l o e ve e o e c colo Fo e c c ve e $_i$ o le $_i$ be pol o ob ed b jo e e o le v $_i$ o e o ve ce e ob e ve ow e c o e e $_j$ $_i$ $_i$ r d ce d ec o o F e o e ob e ve o e c $_i$ e pol o o co de o ed b $_i$ l o $_i$ I ow ollow e e o pol o $_j$ $_i$ $_i$ r c be e e bled o o pol o l o Ob e ve ow e e le o c l o be e e bled o o o pol o l o z we ve

5. *r i g i h r i , h di i i h i ,* ⊔

e ow ow ow o ob $k+$ d ec o o w o $+k$ $+$ $_3$ p ece c e p ece c be e e bled o o **7** o $k+$ pol o l o ce e colo o e ve ce o d ce p o o ve ce e e c o c cl w o $_3$ ve ce ppo e e e c o c cl co e ve ce w colo o $_3$ ele e e be e d ec o o ob ed o b dd c lo e d o l o jo p o ve ce colo ed w colo d ee F d b Ob e ve e be o d o l o w e dpo colo ed d e c l e be o ve ce o colo d w e we c e p ece o lo e c o e e d o l e be o p ece c e e b ee F ce e e e o $\overline{3}$ ve ce w colo we ve e be o p ece o o

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Ob e ve ow e p ece o $_i$ $_i$ colo w e e bled p ope l o pol o $\mathcal{Q}$ l o d ec ed lo d o l co e po d o o e o jo p o ve ce colo ed d e be e d ec o d ced $\mathcal{Q}$ b Co b w we ob 7 d ec o o w o

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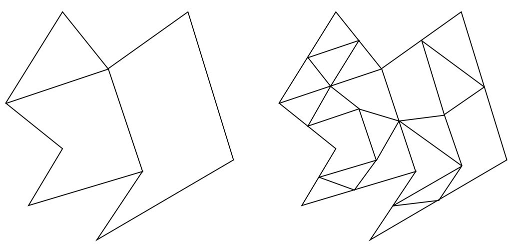

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6. *r i g i h r i h di i i h* $+k$ $+ \overline{3}$ *i h h i i r d r* , 7, , *r* $k+$ *g i i r* ⊔

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pol o c lled ped e e po *p* c e l e e e co ec o o e po co ed e ow ow o ob d ec o o *r h d* pol o w ve ce v k + p ece c e p ece o e e d ec o c be e e bled o o 7 o $k+$ pol o l o e b p ov

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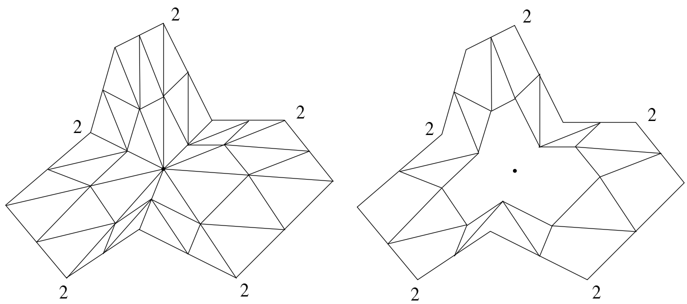

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e be ped pol o d le *p* be po e e o o c e l e e e co ec *p* o po o ll co ed ppo e eve be o ve ce Colo e ve ce o w colo d c w dj ce colo ece ve d e e colo Co ec *p* o ll e ve ce o o ob e o le ow F bd v de e le ob ed o b le e d po o e ed e ow e e e e dele e e ed e co ec *p* o e ve ce o F b o ob d ec o o w + p ece o e o w c l o Ob e ve ll e p ece co ve e o colo e p c be e o ped o o ped pol o l o e e le c l o be e o ped o o o pol o l o e c e w e odd be o ve ce c be do e l w e cep we colo e c l o e ve e o w bo colo d e de l e le o e e de e ple o c e ow F B e p ece o l o we c ob $k+$ d ec o o w k + p ece we ve p oved

8. *r h d g i h r i h di i i h k + i h h i i r d r , 7, , r k+ g i i r* ⊔

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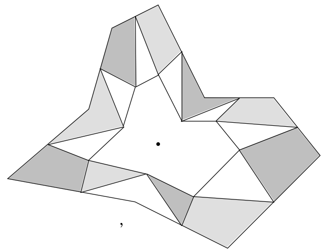

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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 67-80, 2001.

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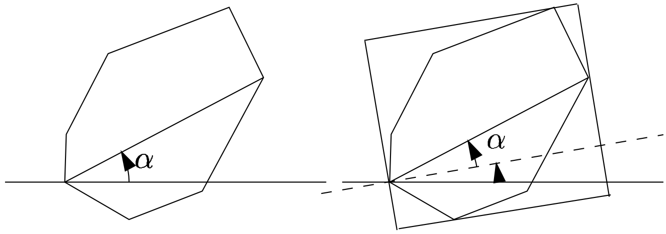

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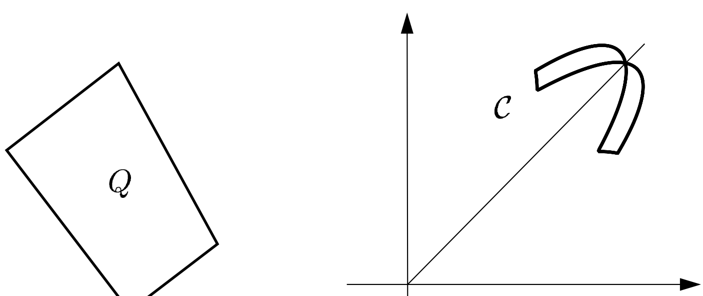

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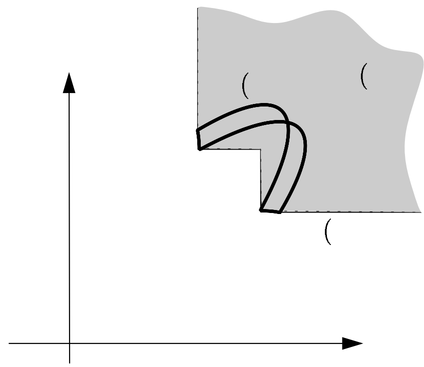

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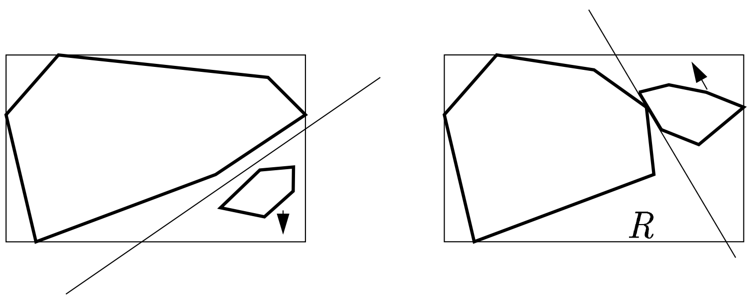

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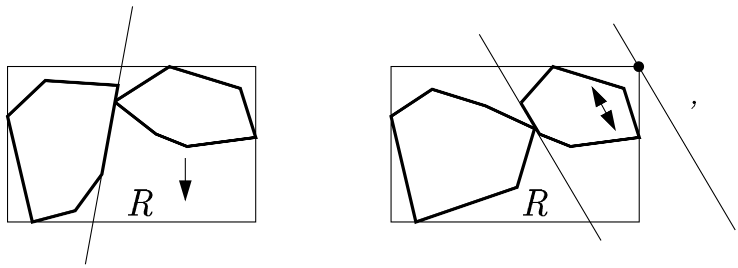

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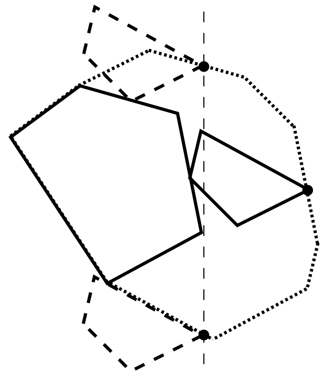

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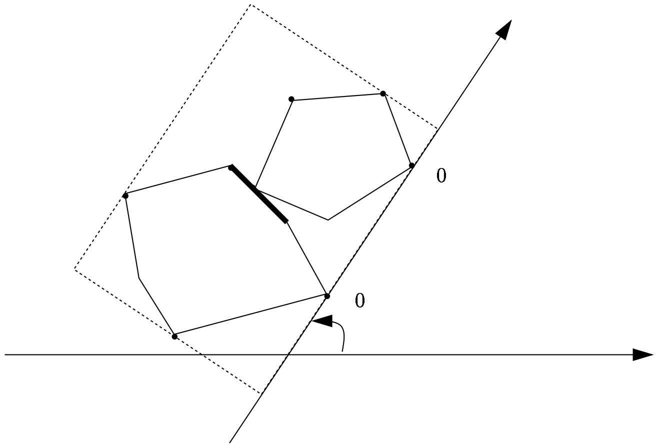

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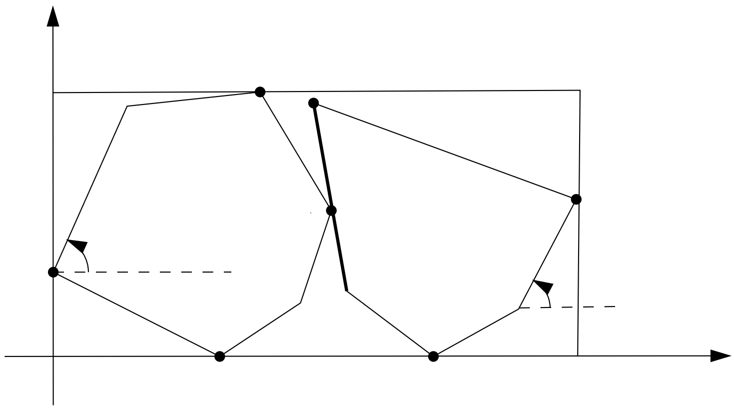

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O h r s l h r l s

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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 81-90, 2001.

be o ve ce d ed e o e ce e e e c be o ^{3}
c eve d e e c c l eve p o e v ewpo p ce o
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ce e co o le w e e o o occl o e co de ed 5
Fo p llel objec e o o p c odel e bo d e
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e o c e o ew w e we co de ce e co o k d
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e c e wo c e e c e ed e d ve ce
o e bo d e o o l k co ve objec l e be o o occl ded
o eve c occ o o e ^2k o e co e po d o
occl ded eve d ppe e v ewpo p ce p o e ce
e ppe bo d o e be o d c o o p c d pe pec ve v ew
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l eb c c ve o co de ee e ec e c c c l ce
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o e o ve p opo ed e c ed o o o opolo c ll d e e
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U e l we ow ow o e wo cop e o o co c o o o ce
e e e ce o k^2 d c o o p c v ew l ec q e
ee cop e ow k^3 lowe bo d o e be o d c pe pec ve
v ew Bo bo d c e ppe bo d o

I e p oceed ve o we d cl ed l ec q e c l o
be ed o eld wo c e lowe bo d o o e be o d

c pe pec ve v ew o p llel objec U o el c o be e ed b d ec ppl c o o o ec q e d e co jec ed lowe bo d o e ope

F ll we c od o co c o l l o p ove e ollow lowe bo d o e be o d c l o e e v ew k^2 e l e o o odel ^2k e o o p c odel d ^3k e pe pec ve odel U o el we e ble o p ove e e lowe bo d e l eb c o o odel

O p pe c ed ollow I e e ec o we p e e e co c o I ec o we ow ow co c o c be ed o eld ll e lowe bo d o e be o v ew e v o odel e o ed bove F ll ec o we e b pe b o e l o o co c o e b de e e c e o lowe bo d e v l d well o o de e e e co c o

e i o t tio

e p e e co c o o ce e w k pol ed o o l co ple le ^2k d c v ew e l e o o odel ee e I dd o o d k e ce e we co c depe d o c l p e e d o ε c be ed c o o eq ed o p od ce ce e w o o p c o pe pec ve v ew d c be e o o ow

Mo e p ec el we p ove e ollow

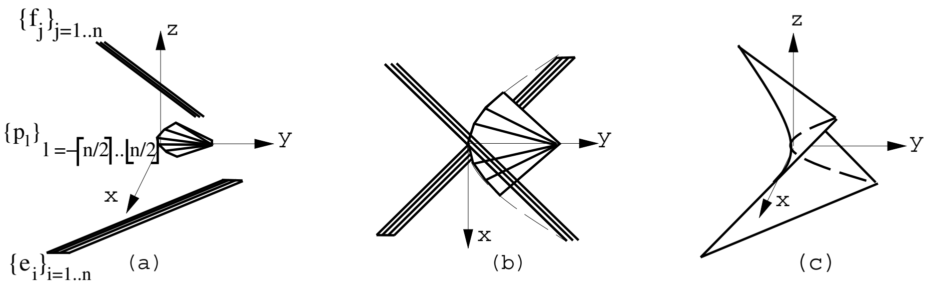
. h i i g h i i h + , i ^2k ri i r di i i , d h r r r di i i h

bo d p o c ll e be po ble

e d d ec o o o o o l o e d ec o d o e ed dow w d c be p e e zed b e p c l l e ed e eo e co e po d o o o p c v ew d ec o o c d ec o e o o o l o e d ec o B e v ewpo ove lo l e ^{3} c e l o we c ob e l l o eo e bo pe pec ve v ew

. pt n

O co c o co o ee d c ele e e *dr* e *d* d e e d ec o o e l o zo l p ed e we co de co ple el c lc l o d e c oo e c v e ll e o o l o e doe o ec e co c o e co ve pol o w ed e c be c e ed o p ope co ve pol ed o e eedle e e o k o zo l p llel l e e eve ll epl ced b ve d lo e ed



i . . T s n n = :) n p s n s n b n n l n n l s n p b) n p $z = \infty$) p b l n s p b l s n n n p s p ss n n

e o be d e co c o o pl ce e ele e o e c co b o o d ed e eedle d ed e eld occl ded eve c be do e b pl c e ε e bo ood o e pe bol c p bolo d e pl ce e d e pl e e ed e o e d e p llel o e l e d o ε w o e eedle e e pl e p llel o d o ε w o e l e Bo l e d belo o ee e c e pl e c lo e p bol ^{2} ^{2} e pol o pl e ze e p bolo d ve ce e c o e o e p bol e e c coo d e o o co c o e ve e ble below

	b	n
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a le . T l n s ns n fix = ——

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e ow p oceed o p ove eo e e ow e c ple o ed e $_i$ f_j p p ve e o occl ded eve o k d $\lceil \frac{}{2} \rceil$ $\frac{}{2}$ e l o ow e co e po d c c l l e o cl de l e o o o l o e d ec o

I o de o p ove e l e e e we od ce e ce $_{i\,j}$ led b l o l e o o o l o e d ec o d p o e d ed e $_i$ d e eedle f_j I ow $_{i\,j}$ pe bol c p bolo d e

Co e eq o o $_{i\,j}$ o e o

$$_{i\,j} \quad _{i\,j}$$

w e e

$$_{i\,j} \quad \Big(\; + \frac{\varepsilon}{}\Big(\frac{}{k} \; - \quad \frac{\varepsilon}{}\Big(\frac{}{k} + -$$

I deed eq o o e o α + pl $_{i\,j}$ co l o l e o o o l o e d ec o e we d $_{i\,j}$ co $_i$ d l l o $_{i\,j}$ co f_j ve po ^{3} o o e l e ppo $_i$ o f_j e e q e l e p o d e l e ppo $_i$ d f_j l e e e ec o o e pl e $_i$ d f_j ee F e e e d ec o o l e be $_{i\,j}$ $_{i\,j}$ e w ll e e ollow p ope o $_{i\,j}$ ce $_{ij}$ co ll e l e l eo l o c e l e co $_i$ f_i d e o o o l o e d ec o $_{i\,j}$ d o l q l e o $_{ij}$ I c e e de w ll ve e l o e ple b co o $_{i\,j}$ $_{i\,j}$ d o ^{3} $_{i\,j}$ d $_{i\,j}$ o ^{3} $_{i\,j}$ e

$$\begin{cases} _{i\,j} & _{i\,j} \\ _{i\,j} & _{i\,j} \\ _{i\,j} & _{i\,j} \end{cases}$$

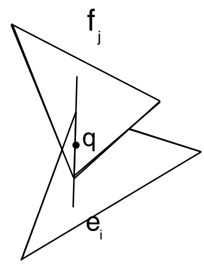
I v ew o o o l c ow be ed ollow ce o p ove e c e e p p e ec e c pe bol c p bolo d $_{i\,j}$ I deed e e ec o p p $_{i\,j}$ co de e l e o p o $_i$ d f_j p ove bo e ple $_i$ f_j d p p d ce eve d pl e e co e po d l e o o o o l o e d ec o o e e e c be o wo c e ec o bec e $_{i\,j}$ pe bol c p bolo d Below we p ove e ollow e dpo o p d p e

$$\begin{cases} p \quad p & _{i\,j} \\ & _{i\,j} \end{cases}$$

I e p e ec $_{i\,j}$ e c l o ce d o p I o ld be cle l o e co e po d l e o e o de ed b lope e d ec o e o c e v l e $\lceil \frac{}{2} \rceil$ $\frac{}{2}$ d e co e po d v ewpo o e e pl e + e ll d c

r e be b ew e eq o o $_{i\,j}$ $_{i\,j}$ $+\delta_{i\,j}$ $\delta_{i\,j}$ w $\delta_{i\,j}$ $\delta_{i\,j}$ ε d le od ce e o o $-$ o e p ^{2} e co p e

$$_{i\,j}\, p \quad ^2 \quad + \delta_{i\,j} \quad \delta_{i\,j}$$
$$\delta_{i\,j} + \delta_{i\,j}$$



i . . T s n l n p ss n p n n n n n n
s b n s n s n pl n s) n)

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Fo we ve $\frac{}{2}$ $+$ d $\frac{t}{2}$ ^{2} $+$ $^2)$
e e o e

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$$- \quad ^2 \quad - \quad + \quad \delta_{i\,j} \quad \delta_{i\,j}$$

$$\frac{}{2} \quad - \quad + \quad \delta_{i\,j} \quad \delta_{i\,j}$$

ce $-$ Bec e d $\delta_{i\,j}$ $\delta_{i\,j}$ ε we ve o o
c o ce o ε ——

$$i\,j \quad \frac{}{2} \quad \varepsilon$$

e e o e we ve $i\,j$ co ple e e p oo o d b e e o
o eo e

e owe o

e b d c o v o lowe bo d co c o ee F e o
p c o l e pl o o o co c o F we eed e l bo e
c c l c ve d ce

. t nd

c o o p c v ewpo c be ep e e ed b o e ed o o e eo
vec o o e we e o l co ce ed bo e d ec o c
e e we e e e be o o e eo coo d e d
e e o vec o c c be ep e e ed b e vec o o
b jec vel b po e pl e I pl e e
eq o

e d ec o o e l e p o $_i$ f_j d p p o c ve
$_{i\,j}$ pl e c po o c ve co e po d o e po o co c
p + p o p p ee F e w e e dpo o
$_{i\,j}$ co e po d o p d p e c ve ll p e e zed
b w P ope ow c e c c c ve
c o e e w ce ee F e o ce o o e $\frac{}{2}$ d o e o
o e $\frac{}{2}$ e co de o l e po o $\frac{}{2}$ o ow o

I o de o e o co pl c ed p e o e c o l e o ec o
d we w o d ec le × c e c c l c ve
co ple el ve e e d ec o d e co ed e v l
e d ec o e c co p e e v l k c
p e e e e v l $\frac{}{2}$ o eve c o ce o
× k × $\lceil\frac{}{2}\rceil$ + $\frac{}{2}$ l l we c co p e
e v l k c e lw pl e

$_{i\,j}$

$-$

$_{i\,j}$

$_{i\,j}$ $_s$ $-$

$_{i\,j}$ $_s$ $-$

d le d e co d o o ollow
o d F ll we ow e e ec o o e $_{i\,j}$ w e
e ll d c e c e e o e e b e v l o co
c e $_{i\,j}$ do o p w e e ec e ec le ×

I e c c l c ve co ple el ve e k e d ec o
e co ed e v l k e d ec o d do o p w e
e ec e ec le k × k ll e o e
c c l c ve loo F e b w e de c lo e e
de

o e l o ee o w e d ec o e pe pec ve odel
e c c l ce loo l e F e c I deed we c l l elec e v l
k Y k d k o e e le c e c c l ce
do o p w e e ec k × Y k × k d do o c o e op
d bo o ce o e bo e co c o o w d d o ed
po ble o e p e d Y c o o d d b e o
o e e v l + o o e l e e o e o dep c ed o
F e b o e o o p c odel d c o e pe pec ve odel

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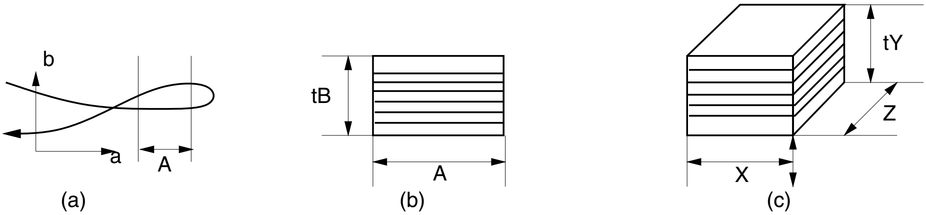
n s s n s n yp b l p b l pl n
nfin y n n b s n y yp b l s n pl n 6] ll n
n n



i .3.) l n s n l b) T l s n p l) T l s s n p n s s n ly y n p sp l

. n b n t t p d

Fo e o o p c odel pl ce wo cop e o e ce e o F e o e be e e o e o e e o o o le o bo e d ble l o o vo d coll o F e B e e l o e d c o e p ev o ec o e v ewpo p ce e c c l c ve ove l p d l e o e od w c we c ll e *k d i i* e od l e d ed p ove o ce e eld ^2k ^{2} k^2 c c l eve

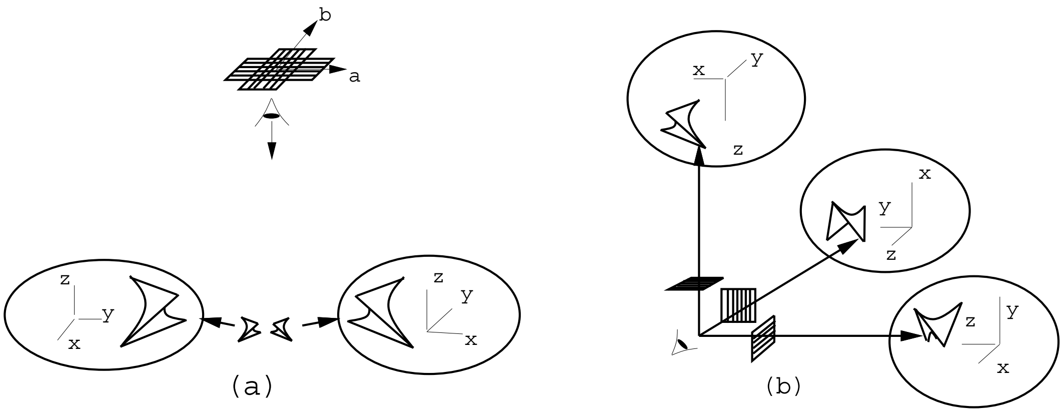
. n b n t p p t d

Fo e pe pec ve odel pl ce e v ew po ve e o c be w c e l l e de d pl ce ee cop e o e ce e o F e e ee dj ce ve ce F e b w e o e o pe ed c w e c c l ce w ll ove l p d l e o eld ^2k ^{3} k^3 c c l eve e od w c we c ll e *k ri i i* e od l o l e d ed

. n b tt

Fo l o e e v ew we od e co c o b epl c e ed e o e d b *k* eedle $_i$ e e bo ood o *k* I o ld be cle ll e eve co e po d o $_i$ f_j d *p p* e l o e e eve d e e e bove ow deed o e k^2 l o e e eve e occl ded d c be ob e ved e v ewpo ove lo e e pl e + o l e ^{3} c e l o e ce e o pe pec ve v ew e bo d k^2 o e be o l o e e v ew e l e o o odel

e bo d e e d ll o ^2k o e be o l o e e v ew e o o p c odel d o ^3k o e be o l o e e v ew e pe pec ve odel b e e ew l pl c o e od e p ev o wo ec o



i . . n s n s b n pl x y p n sp p ns n) p l b) p sp l

A o t e e e ie

e e e ee d o de e e c e o co c o e ce o e d e copl e eedle e e e d o pol ed d e pol o d o pol ed o e e ow o e o e ec o e l

o co pe e o e c e ce o e d e copl we c od ce pe b o o p e e ed e co ve po o depe d o ll po ve p e e b p ed e $_i$ e pl e ^{2}

e eedle e e e b we c epl ce e lle e dpo o e c eedle b ll le co e e dpo w o e de e lle p e e $_2$ eedle ve lo d e ed o B c oo pp op e o e o o e e e ed we c e f_j v ble o e co o w e v ewed o ll e e ve c l d ec o o e e p pe e e ve ce o e le e b o f_j o be f_j f_j $+$ $_2$ d f_j $+$ $_2$ $+$ $_2^{2}$ ^{2}

p e e ed e pol o e pol ed o e c e e o co ve pol ed o w o e p e o b e co ve ll o e p w ed e o le $_3$ w o e e dpo e $_3$ d ^{3}

e e e od c o o e doe o ec e ve c o e p oo c be o ll c ec ed b e ollow pp o c c ec e e o $_2$ o $_3$ ppe e p o c ll do e o q o w c we e e po ve o e ve we do o el o q o be e c l c e z p ope o e e e ce o k k d k bove e e o e e bo d e v l d o c e l ll v l e o $_2$ d $_3$

Re e e e

G *v* *s* p n l
y n C y sp p s: n n n n s y n
s l s *g* *g* *S* *s* *s* *d* *g* : 5– 0
lp n s n n l p s n n s
n n b s p ly l s n s *G*
: 5– 5
b s ll l n n n n n l n n
s l ps p ly l s n n *P* *M* *S* *M* *S* *s* *s*
g *s* 000
5 l n n n C y s b l y l s n n sp p
V *s* 5): – 60 0
6 n y n T n b s x s p ll l bj s *g* *s* *v* : –

Problems and Results around the Erdős–Szekeres Convex Polygon Theorem*

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1 Introduction

Eszter Klein's theorem claims that among any 5 points in the plane, no three collinear, there is the vertex set of a convex quadrilateral. An application of Ramsey's theorem then yields the classical Erdős–Szekeres theorem [19]: *For every integer $n \geq 3$ there is an N_0 such that, among any set of $N \geq N_0$ points in general position in the plane, there is the vertex set of a convex n-gon.* Let $f(n)$ denote the smallest such number.

Theorem 1 ([20,44]).

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-5}{n-2} + 2 \ .$$

A very old conjecture of Erdős and Szekeres is that the lower bound is tight:

Open Problem 1. *For every $n \geq 3$, $f(n) = 2^{n-2} + 1$.*

Similarly, let $f_d(n)$ denote the smallest number such that, in any set of at least $f_d(n)$ points in general position in Euclidean d-space, there is the vertex set of a convex polytope with n vertices, that is, *n points in convex position.* A simple projective argument [47] shows that $f_d(n) \leq f(n)$. It is conjectured by Füredi [22] that $f_d(n)$ is essentially smaller if $d > 2$, namely that $\log f_d(n) = O(n^{1/(d-1)})$. A lower bound that matches this conjectured upper bound was given recently in [33]. On the other hand, Morris and Soltan [34] contemplate about an exponential lower bound on $f_d(n)$.

In this paper we survey recent results and state some open questions that are related to Theorem 1. In particular, we consider "homogeneous", "partitional", and "modular" versions of the Erdős–Szekeres theorem. We will discuss the question whether empty convex polygons (and then how many of them) can be found among N points in the plane. We will also describe how the convex

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position condition can be strengthened or relaxed in order to arrive at well-posed questions, and present the results obtained so far.

For further aspects of the Erdős–Szekeres theorem we refer to the very recent and comprehensive survey article [34].

2 Homogeneous Versions

From now on we assume that $X \subset \mathbb{R}^2$ is a finite set of points in general position. We assume further that X has N elements. By the Erdős–Szekeres theorem, any subset of X of size $f(n)$ contains the vertices of a convex n-gon. As a fixed n-set is contained in $\binom{N}{f(n)-n}$ subsets of size $f(n)$, a positive fraction of all the n-tuples from X are in convex position. This is a well-known principle in combinatorics. Maybe one can say more in the given geometric situation, for instance, the many convex position n-tuples come with some structure. The following theorem, due to Bárány and Valtr [7], shows that these n-tuples can be chosen homogeneously:

Theorem 2 ([7]). *Given $n \geq 4$, there is a constant $C(n)$ such that for every $X \subset \mathbb{R}^2$ of N points in general position the following holds. There are subsets $Y_1, \ldots, Y_n$ of X, each of size at least $C(n)N$ such that for every transversal $y_1 \in Y_1, \ldots, y_n \in Y_n$, the points $y_1, \ldots, y_n$ are in convex position.*

We call this result the "homogeneous" Erdős–Szekeres theorem. The proof in [7] is based on another homogeneous statement, the so called same type lemma. We state it in dimension d, but first a definition: Two n-tuples $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are said to be of the *same type* if the orientations of the simplices $x_{i_1}, \ldots, x_{i_{d+1}}$ and $y_{i_1}, \ldots, y_{i_{d+1}}$ are the same for every $1 \leq i_1 < i_2 < \ldots < i_{d+1} \leq n$.

Theorem 3 ([7]). *Given $d \geq 2$ and $k \geq d+1$, there is a constant $C(k,d)$ such that for all finite sets $X_1, \ldots, X_k \subset \mathbb{R}^d$ of points such that $\cup_1^k X_i$ is in general position the following holds. For every $i = 1, \ldots k$, the set X_i contains a subset Y_i of size at least $C(k,d)|X_i|$ such that all transversals $y_1 \in Y_1, \ldots, y_k \in Y_k$ are of the same type.*

The proof is based on the center-point theorem of Rado [7], or on Borsuk's theorem [37]. It uses a reformulation of the definition of same type: all transversals of $Y_1, \ldots, Y_k$ are of the same type if no hyperplane meets the convex hulls of any $d+1$ of these sets. The same type lemma implies the homogeneous version of the Erdős–Szekeres theorem in the following way. Choose $k = f(n)$ and partition $X \subset \mathbb{R}^2$ by vertical lines, say, into sets $X_1, \ldots, X_k$ of almost equal size. Apply the same type lemma to them. All transversals of the resulting subsets $Y_1, \ldots, Y_k$ are of the same type. Fix a transversal $y_1, \ldots, y_k$. As $k = f(n)$, the Erdős–Szekeres theorem implies that some n points of this transversal, $y_{j_1}, \ldots, y_{j_n}$, say, are in convex position. Then by the same type lemma, all transversals of $Y_{j_1}, \ldots, Y_{j_n}$ are in convex position.

This proof gives a doubly exponential lower bound for $C(n)$. An alternative proof, with a better bound for $C(n)$ was found by Solymosi [39]. A sketch of Solymosi's neat argument goes as follows. As we have seen above, a positive fraction of the $2n$ element subsets of X are in convex position. Write such a $2n$ element subset as $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ with the points coming in this order on the boundary of their convex hull. Choose $a_1 = x_1, a_2 = x_2, \ldots, a_n = x_n$ so that the number of possible convex extensions $a_1, y_1, a_2, y_2, \ldots, a_n, y_n$ is maximal. Averaging shows that the number of such extensions is at least const$\binom{N}{n}$ where $|X| = N$. A simple geometric argument explains that the possible y_is all lie in the triangle T_i formed by the lines through the pairs of points (a_{i-1}, a_i), (a_i, a_{i+1}) and (a_{i+1}, a_{i+2}). It is not hard to check then that $Y_i = X \cap T_i$ $(i = 1, \ldots, n)$ satisfies the requirements.

This proof gives $C(n) \approx 2^{-16n^2}$, while the lower bound for $f(n)$ shows that $C(n)$ is at least 2^{n-2}. Better bounds are available for $n = 4, 5$ [7]: $C(4) \geq 1/22$ and $C(5) \geq 1/352$. The reader is invited to prove or improve these bounds.

Pach [37] uses the same type lemma to prove a homogeneous version of Caratheodory's theorem that was conjectured in [6]: Given $X_i \subset \mathbb{R}^d$ $(i = 1, \ldots, d+1)$, there is a point $z \in \mathbb{R}^d$ and there are subsets Z_i, each of size $c_d|X_i|$ at least $(i = 1, \ldots, d+1)$, such that the convex hull of each transversal $z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}$ contains the point z. (Here c_d is a constant depending only on d.) Pach's nice argument uses, besides the same type lemma, a quantitative version of Szemerédi's regularity theorem.

We expect that the same type lemma will have many more applications. Also, several theorems from combinatorial convexity extend to positive fraction or homogeneous versions. For instance, a positive fraction Tverberg theorem is proved in [7]. One question of this type concerns Kirchberger's theorem [14]. The latter says that finite sets $A, B \subset \mathbb{R}^d$ can be separated by a hyperplane if and only if for every $S \subset A \cup B$ of size $d+2$ there is a hyperplane separating $A \cap S$ and $B \cap S$. This suggests the following question:

Open Problem 2. *Let $A, B \subset \mathbb{R}^d$ be finite sets, each of size N, with $A \cup B$ in general position. Assume that for $(1-\varepsilon)$ fraction of the $\binom{2N}{d+2}$ $(d+2)$-tuples $S \subset A \cup B$ there is a hyperplane separating $A \cap S$ from $B \cap S$. Does it follow then that there are subsets $A' \subset A$ and $B' \subset B$ that are separated by a hyperplane and $|A'|, |B'| \geq (1 - g(\varepsilon))N$ with $g(\varepsilon)$ tending to zero as $\varepsilon \to 0$?*

Partial results in this direction are due to Attila Pór [40]. One word of caution is in place here: the condition $g(\varepsilon) \to 0$ is important since by the ham-sandwich theorem (or Borsuk's theorem, if you like) any two finite sets A, B in $\mathbb{R}^d$ can be simultaneously halved by a hyperplane H. Then half of A is on one side of H while half of B is on the other side.

3 Partitional Variants

Let P be any set of points in general position in the plane. Let C_1, C_2 be subsets of P, each in convex position. We say that the *convex polygons* C_1 and C_2 are *vertex disjoint* if $C_1 \cap C_2 = \emptyset$. If, moreover, their convex hulls are also disjoint, we simply say that the two polygons are *disjoint*. A polygon is called *empty* if its convex hull does not contain any point of P in its interior.

Eszter Klein's theorem implies that P can be partitioned into vertex disjoint convex quadrilaterals plus a remainder set of size at most 4. The following result answers a question posed by Mitchell.

Theorem 4 ([30]). *Let P be any set of $4N$ points in general position in the plane, N sufficiently large. Then there is a partition of P into N vertex disjoint convex quadrilaterals if and only if there is no subset A of P such that the size of A is odd but the the size of $A \cap C$ is even for every convex quadrilateral C.*

There is also an $N \log N$-time algorithm [30] which decides if such a partition exists. The following problem seems to be more difficult.

Open Problem 3. *Is there a fast algorithm which decides if a given set of $4N$ points in general position in the plane admits a partition into disjoint convex quadrilaterals?*

For $k \geq 3$ the *Ramsey-remainder* $rr(k)$ was defined by Erdős et al. [21] as the smallest integer such that any sufficiently large set of points in general position in the plane can be partitioned into vertex disjoint polygons, each of size $\geq k$, and a remaining set of size $\leq rr(k)$. Thus, $rr(k) < f(k)$ for every k. In particular, $rr(3) = 0$ and $rr(4) = 1$. Partial results on $rr(k)$ in general were proved in [21]. It is known, for example, that $rr(k) \geq 2^{k-2} - k + 1$. The solution of the following problem could make an essential step to settle Problem 1, see [21].

Open Problem 4. *Is it true that $rr(k) = 2^{k-2} - k + 1$?*

There is no Ramsey-remainder in higher dimensions. The following result is due to Károlyi [30].

Theorem 5. *Let $k > d \geq 3$. If N is sufficiently large, then every set of N points in general position in $\mathbb{R}^d$ can be partitioned into subsets of size at least k each of which is in convex position.*

The main observation here is that, for large enough N, every point of P belongs to some k-element subset which is in convex position.

A problem in close relation to Problem 3 is the following. Given natural numbers k and n, let $F_k(n)$ denote the maximum number of pairwise disjoint empty convex k-gons that can be found in every n-element point set in general position in the plane. The study of this function was initiated in [27]. Horton's result mentioned in Section 5 implies $F_7(n) = 0$ for every n. Thus, the interesting functions are F_4, F_5 and F_6. Nothing is known about F_6, in fact Problem 6 is equivalent to asking whether $F_6(n) > 0$ for some n. Since every 5-point set determines an empty convex quadrilateral, obviously $F_4(n) \geq \lfloor n/5 \rfloor$. Similarly, it follows from a result of Harborth [23] that $F_5(n) \geq \lfloor n/10 \rfloor$ for every n.

The non-trivial lower bound $F_4(n) \geq \lfloor 5n/22 \rfloor$ is presented in [27], based on the following observation. Suppose P is any set of $2m+4$ points in general position in the plane. Then there is a partition of the plane into 3 convex regions such that one region contains 4 points of P in convex position, and the other regions contain m points of P each. There is no counterpart of this lemma for pentagons, and in fact no lower bound is known about F_5 beyond what is said above. As for F_4, an even stronger lower bound $F_4(n) \geq (3n-1)/13$ has been proved for an infinite sequence of integers n.

Concerning upper bounds, a construction in [27] shows that $F_5(n) \leq 1$ if $n \leq 15$. It is not too difficult to prove that $F_5(n) < n/6$, but no nontrivial upper bound is known for $F_4(n)$ in general.

For any positive integer n let $F(n)$ denote the smallest integer such that every set of n points in general position in the plane can be partitioned into $F(n)$ empty convex polygons, with the convention that point sets consisting of at most two points are always considered as empty convex polygons. Urabe [45] proved $\lceil (n-1)/4 \rceil \leq F(n) \leq \lceil 2n/7 \rceil$. The upper bound follows from the fact that every 7-point set can be partitioned into an empty triangle and an empty convex quadrilateral.

An improved upper bound $F(n) \leq \lceil 5n/18 \rceil$ is presented in [27] along with an infinite sequence of integers n for which also $F(n) \leq (3n+1)/11$.

An other function $H(n)$ was also introduced in [45] as the smallest number of vertex disjoint convex polygons into which any n-element point set can be partitioned in the plane. An application of Theorem 1 gives that the order of magnitude of this function is $n/\log n$.

Finally we mention that the functions F and H can be naturally defined in any dimension; denote the corresponding functions in d-space by F_d and H_d. Urabe [46] proves that $\Omega(n/\log n) \leq F_3(n) \leq \lceil 2n/9 \rceil$ and that $H_3(n) = o(n)$. The proof technics of [45] coupled with the bounds given in Section 1 on f_d in fact yield $\Omega(n/(\log^{d-1} n) \leq F_d(n) \leq O(n/\log n)$.

4 Matrix Partitions

Assume $X_1, \ldots, X_n$ in $\mathbb{R}^d$, are pairwise disjoint sets, each of size N, with $\cup X_i$ in general position. A matrix partition, or μ-partition for short, of the X_is with m columns is the partition $X_i = \cup_{k=1}^{m} M_{ik}$ for $i = 1, \ldots, n$ if $|M_{ik}| = |M_{jk}|$ for every $i, j = 1, \ldots, n$ and every $k = 1, \ldots, m$. In other words, a μ-partition of $X_1, \ldots, X_n$ with m columns is an $n \times m$ matrix M whose (i,k) entry is a subset M_{ik} of X_i such that row i forms a partition of X_i and the sets in column k are of the same size. Gil Kalai asked [28] whether the homogeneous Erdős–Szekeres theorem admits a partitioned extension:

Open Problem 5. *Show that for every $n \geq 4$ there is an integer $m = g(n)$ such that for every finite set $X \subset \mathbb{R}^2$ of N points in general position there is a subset $X_0 \subset X$, of size less than $f(n)$, (this is the Erdős–Szekeres function from*

Section 1), and there exists a partition of $X \setminus X_0$ *into sets* $X_1, \dots, X_n$ *of equal size such that the following holds. The sets* $X_1, \dots, X_n$ *admit a* μ*-partition* M *with* m *columns so that every transversal* $x_1 \in M_{1k}, x_2 \in M_{2k}, \dots, x_n \in M_{nk}$ *is in convex position, for all* $k = 1, \dots, m$.

By the homogeneous version one can choose the sets for the first column of a μ-partition, each of size $C(n)N/n$, then for the second, third, etc columns from the remaining part of X, but this would result in a suitable μ-partition with too many, namely $\log N$, columns. The remainder set X_0 is needed for two simple reasons: when N is smaller than $f(n)$ there may not be a convex n-gon at all, and when N is not divisible by n.

Partial solution to Problem 2 is due to Attila Pór [41]. He first proved a partitioned extension of the same type lemma. To state this result we define the sets $Y_1, \dots, Y_n \subset \mathbb{R}^d$ with $n \geq d+1$ *separated* if every hyperplane intersects at most d sets of the convex hulls of $Y_1, \dots, Y_n$. As we mentioned in Section 2, the sets $Y_1, \dots, Y_n$ are separated if and only if every transversal $y_1 \in Y_1, \dots, y_n \in Y_n$ is of the same type.

Theorem 6 ([41]). *For all natural numbers* n, d *with* $n \geq d+1$ *there is a natural number* $m = m(n,d)$ *such that if finite sets* $X_1, \dots, X_n \subset \mathbb{R}^d$ *have the same size and* $\cup_1^n X_i$ *is in general position, then there exists a* μ*-partition with* m *columns such that the sets* $M_{1k}, \dots, M_{nk}$ *in every column are separated.*

This is exactly the partitioned version of the same type lemma. The proof is based on a clever induction argument and a third characterization for sets $Y_1, \dots, Y_n$ being separated. The result is used by A. Pór [41] to solve the first interesting case, $n = 4$ of Problem 2.

Theorem 7 ([41]). *Assume* $X \subset \mathbb{R}^2$ *is a finite set of* N *points in general position. Then there is an* $X_0 \subset X$ *of size at most 4, and a partition of* $X \setminus X_0$ *into sets* X_1, X_2, X_3, X_4 *of equal size such that they admit a* μ*-partition* M *with 30 columns so that every transversal* $x_1 \in M_{1k}, \dots, x_4 \in M_{4k}$ *is in convex position.*

The proof starts by cutting up X into four sets of almost equal size by vertical lines, say. Then the same type lemma (matrix partition version) is applied to these four sets giving a matrix partition with few columns. The columns are of two types: either every transversal is a convex quadrangle and there is nothing to do, or every transversal is a triangle with the fourth point inside it. In the latter case one has to partition the column further. This can be done with a topological argument: the interested reader should consult the paper [41]. The method does not seem to work for $n \geq 5$, apparently new ideas are needed.

5 Empty Convex Polygons

For a long time it had been conjectured that every sufficiently large point set, in general position in the plane contains the vertex set of an *empty* convex n-gon, that is, n points which form the vertex set of a convex polygon with no other point of the set in its interior. Harborth [23] showed that every 10-element point set determines an empty convex *pentagon*, and that here 10 cannot be replaced by any smaller number. Finally, in 1983 a simple recursive construction of arbitrarily large finite point sets determining no empty convex *heptagons* was found by Horton [24]. The corresponding problem for *hexagons* is still open:

Open Problem 6. *Is it true that every sufficiently large set of points in general position in the plane contains the vertex set of an empty convex hexagon?*

We strongly believe that the answer is yes, but there is no proof in sight.

Several algorithms had been designed [4,15,36] to determine if a given set of points contains an empty 6-gon, and to construct large point sets without any empty hexagon. The current world record, a set of 26 points that does not contain an empty convex 6-gon was discovered by Overmars et al. [36] in 1989.

A surprising number of questions can be related to this seemingly particular problem. The first one, due to Solymosi [43], relates it to a Ramsey type problem for geometric graphs. A *geometric graph* is a graph drawn in the plane such that the vertices are represented by points in general position while the edges are straight line segments that connect the corresponding vertices.

Open Problem 7. *Let G be a complete geometric graph on n vertices whose edges are colored with two different colors. Assume that n is sufficiently large. Does it follow then that G contains an empty monochromatic triangle?*

Were the answer to this question negative, it would imply that there are arbitrarily large point sets without an empty convex 6-gon. For assume, on the contrary, that every sufficiently large point set contains such an empty polygon. Color the edges of the corresponding complete geometric graph with two colors, it induces a coloring of the edges that connect the vertices of the empty 6-gon. It follows from Ramsey's theorem that this two-colored graph on 6 vertices contains a monochromatic triangle (which is also empty), a contradiction.

An other related problem has been studied recently by Hosono et al. [26]. Let P denote a simple closed polygon together with its interior. A *convex subdivision* of P is a 2-dimensional cell complex in the plane whose vertex set coincides with the vertex set of P, whose body is P, and whose faces are all convex polygons. Denote by $F'(n)$ the smallest integer for which any set of n points in general position in the plane can be connected with a closed simple polygon that admits a convex subdivision with at most $F'(n)$ faces. Since each face in a convex subdivision is an empty convex polygon, it follows from Horton's construction that $F'(n) \geq n/4$ for an infinite sequence of n. It is proved for every n in [26] where an upper bound $F'(n) \leq \lceil 3n/5 \rceil$ is also presented.

Open Problem 8. *Is it true that $F'(n) \geq (n-2)/3$?*

A negative answer would give an affirmative solution to the empty hexagon problem.

Essential combinatorial properties of Horton's construction were studied and extended into higher dimensions by Valtr [47], resulting in constructions that yield the following general result. Denote by $h(d)$ the largest integer h with the following property: every sufficiently large point set in general position in $\mathbb{R}^d$ contains an *h-hole*, that is, h points which are vertices of an empty convex d-polytope. Thus, $5 \leq h(2) \leq 6$.

Theorem 8 ([47]). *The integer $h(d)$ exist for any $d \geq 2$ and satisfies*

$$2d + 1 \leq h(d) \leq 2^{d-1}(P_{d-1} + 1) \ ,$$

where P_i denotes the product of the first i positive prime numbers.

It is also known that $h(3) \leq 22$.

We close this section by turning back to the plane: there are certain nontrivial classes of point sets where large empty convex polygons can be found. For example, if every triple in the point set determines a triangle with *at most one* point in its interior, then it is said to be *almost convex*.

Theorem 9 ([32]). *For any $n \geq 3$, there exists an integer $K(n)$ such that every almost convex set of at least $K(n)$ points in general position in the plane determines an empty convex n-gon. Moreover, we have $K(n) = \Omega(2^{n/2})$.*

This result has been extended recently by Valtr [50] to point sets where every triple determines a triangle with at most a fixed number of points in its interior. It also must be noted that Bisztriczky and Fejes Tóth [10] proved the following related result.

Theorem 10. *Let l, n denote natural numbers such that $n \geq 3$. Any set of at least $(n-3)(l+1)+3$ points in general position in the plane, with the property that every triple determines a triangle with at most l of the points in its interior, contains n points in convex position. Namely, its convex hull has at least n vertices, and in this respect this bound cannot be improved upon.*

6 The Number of Empty Polygons

Let $X \subset \mathbb{R}^2$ be a set of N points in general position, and write $g_n(X)$ for the number of empty convex n-gons with vertices from X. Of course, $n \geq 3$. Define $g_n(N)$ as the minimum of $g_n(X)$ over all planar sets X with N points in general position. Horton's example shows that $g_n(N) = 0$ when $n \geq 7$. Problem 6 is, in fact, to decide whether $g_6(N) = 0$ or not.

The first result on $g_n(N)$ is due to Katchalski and Meir [29] who showed $g_3(N) \leq 200N^2$. In Bárány and Füredi [5] lower and upper bounds for $g_n(N)$ are given. The lower bounds are:

Theorem 11 ([5]).

$$g_3(N) \geq N^2 - O(N \log N) \ ,$$
$$g_4(N) \geq \frac{1}{2}N^2 - O(N \log N) \ ,$$
$$g_5(N) \geq \lfloor \frac{N}{10} \rfloor \ .$$

The last estimate can be easily improved to $g_5(N) \geq \lfloor \frac{N-4}{6} \rfloor$.

Of these inequalities, the most interesting is the one about g_3. Its proof gives actually more than just $g_3(N) \geq N^2(1 + o(1))$. Namely, take any line ℓ and project the points of X onto ℓ. Let $z_1, \ldots, z_N$ be the projected points on ℓ in this order, and assume z_i is the projection of $x_i \in X$. We say that pair z_i, z_j supports the empty triangle x_i, x_k, x_j if this triangle is empty and $i < k < j$. Now the proof of the lower bound on $g_3(N)$ follows from the observation that all but at most $O(N \log N)$ pairs z_i, z_j support at least two empty triangles. (This fact implies, further, the lower bound on g_4 as well.) It is very likely that a small but positive fraction of the pairs supports three or more empty triangles but there is no proof in sight. If true, this would solve the next open problem in the affirmative:

Open Problem 9. *Assume X is a finite set of N points in general position in $\mathbb{R}^2$. Show that $g_3(N) \geq (1 + \varepsilon)N^2$ for some positive constant ε.*

The upper bounds from [5] have been improved upon several times, [48], [16], and [8]. The constructions use Horton sets with small random shifts. We only give the best upper bounds known to date [8].

Theorem 12.

$$g_3(N) \leq (1 + o(1))1.6195...N^2 \ ,$$
$$g_4(N) \leq (1 + o(1))1.9396...N^2 \ ,$$
$$g_5(N) \leq (1 + o(1))1.0205...N^2 \ ,$$
$$g_6(N) \leq (1 + o(1))0.2005...N^2 \ .$$

It is worth mentioning here that the function $g_n(X)$ satisfies two linear equations. This is a recent discovery of Ahrens et al. [1] and Edelman-Reiner [17]. Since the example giving the upper bounds in the last theorem is the same point set X and $g_7(X) = 0$, only two of the numbers $g_n(X)$ $(n = 3, 4, 5, 6)$ have to be determined.

There is a further open problem due to the first author, that appeared in a paper by Erdős [18]. Call the degree of a pair $e = \{x, y\}$ (both x and y coming from X) the number of triples x, y, z with $z \in X$ that are the vertices of an empty triangle, and denote it by $\deg(e)$.

Open Problem 10. *Show that the maximal degree of the pairs from X goes to infinity as the size of X, $N \to \infty$.*

The lower bound on g_3 implies that the average degree is at least $6 + o(1)$ in the following way. Write T for the set of triples from X that are the vertices of an empty triangle. We count the number, M, of pairs (e, t) where $t \in T$, $e \subset T$ and e consists of two elements of X in two ways. First $M = \sum \deg(e)$ the sum taken over all two-element subsets of X. Secondly, as every triangle has three sides, $M = 3|T| = 3g_3(X) \geq (3 + o(1))N^2$ from the lower bound on $g_3(N)$, showing indeed that the average degree is at least $6 + o(1)$.

We show next that the maximal degree is at least 10 when N is large enough, a small improvement that is still very far from the target. Choose first a vertical line ℓ_1 having half of the points of X on its left, the other half on its right. (Throw away the leftmost or rightmost point if N is odd.) Then choose a line ℓ_2, by the ham-sandwich theorem, halving the points on the left and right of ℓ_1 simultaneously (throwing away, again, one or two points if necessary). We have now four sectors, S_1, S_2, S_3, S_4 each containing m points from X with $m = \lfloor N/4 \rfloor$. (S_1, S_4 are on the left of ℓ_1 and S_1, S_2 are below ℓ_2, say.) Let $e = \{x, y\}$ with $x, y \in X$ and define $\deg(e; S_i)$ as the number of points $z \in X \cap S_i$ such that $\{x, y, z\} \in T$. The observation following the lower bounds for g_n gives that, when $e = \{x, y\}$ with $x \in X \cap S_1$ and $y \in X \cap S_2$, then for all but at most $O(m \log m)$ of the possible pairs $\deg(e; S_1 \cup S_2) \geq 2$, so

$$\sum_{x \in S_1} \sum_{y \in S_2} \deg(\{x, y\}; S_1 \cup S_2) \geq (2 + o(1))m^2.$$

On the other hand,

$$\sum_{x \in S_1} \sum_{y \in S_2} \deg(\{x, y\}; S_1 \cup S_2) = \sum_{x, z \in S_1} \deg(\{x, z\}; S_2) + \sum_{y, z \in S_2} \deg(\{y, z\}; S_1).$$

The analogous identities and inequalities for pairs in $S_2 \times S_3$, $S_3 \times S_4$, and $S_4 \times S_1$ together yield that

$$\sum_{i=1}^{4} \sum_{x, y \in S_i} \deg(\{x, y\}; S_{i-1} \cup S_{i+1}) \geq (8 + o(1))m^2,$$

where $i + 1$ and $i - 1$ are to be taken modulo 4. This means that, in at least one of the sectors, the average degree of a pair is at least $4 + o(1)$ in the neighboring two sectors. As we have seen, the average degree of a pair is at least $6 + o(1)$ within each sector. This proves the claim.

7 The Modular Version

Bialostocki, Dierker, and Voxman [9] proposed the following elegant "modular" version of the original problem.

Open Problem 11. *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B(n, p)$ such that every set of $B(n, p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is* $0 \bmod p$.

Bialostocki et al. proved this conjecture for every $n \geq p+2$. Their proof goes as follows. Assume, for technical simplicity, that $n = p+2$. Choose an integer m that is very large compared to n. Consider a set P of $f(m)$ points in general position in the plane, by Theorem 1 it contains an m-element set S in convex position. Associate with every triple $\{a, b, c\} \subseteq S$ one of the p colors $0, 1, 2, \ldots, p-1$; namely color i if triangle abc contains i points of P in its interior modulo p. As a consequence of Ramsey's theorem we can select an n-element subset S' of S all of whose triples are of the same color, given that m is sufficiently large. Consider any triangulation of the convex hull of S', it consists of p triangles. Consequently, the number of points inside this convex n-gon is divisible by p.

This proof implies a triple exponential upper bound on $B(n,p)$, a bound which was later improved essentially by Caro [12], but his proof also relied heavily on the assumption $n \geq p+2$. Recently the conjecture was proved in [32] for every $n \geq 5p/6 + O(1)$. A key factor in this improvement is Theorem 9.

The situation changes remarkably in higher dimensions. For example, a 3-polytope with 5 vertices admits two essentially different triangulations: one into two simplices and an other into three simplices. Based on this observation Valtr [49] proved the following result.

Theorem 13. *For any $n \geq 4$ and $p \geq 2$, there exists an integer $C(n,p)$ such that every set of $C(n,p)$ points in general position in 3-space determines a convex polytope with n vertices such that the number of points in its interior is* $0 \bmod p$.

Indeed, let P be any sufficiently large set of points in general position in 3-space. As in the planar case, we can use the Erdős–Szekeres theorem and then Ramsey's theorem to find at least n and not less than 5 points in convex position such that every tetrahedron determined by these points contains the same number of points, say i, in its interior modulo p. Consider any 5 of these points and triangulate their convex hull in two different ways: first into two tetrahedra, then into three tetrahedra. It follows that $2i \equiv 3i$, and thus $i \equiv 0 \pmod{p}$.

The same argument can be used to extend Theorem 9, and also its generalization by Valtr, to 3-space:

Theorem 14. *Given any natural numbers k and $n \geq 3$, there exists an integer $K_3(k,n)$ such that the following holds. Every set of at least $K_3(k,n)$ points in general position in 3-space, with the property that any tetrahedron determined by these points contains at most k points in its interior, contains an n-hole.*

Similar results are proved also in every odd dimension. First we recall the following strengthening of the Erdős–Szekeres theorem, which seems to be folklore. See [13] or [11, Proposition 9.4.7] for a proof.

Theorem 15. *Let $d \geq 2$. For every $n \geq d+1$ there is an integer $N_d(n)$ such that, among any set of $N \geq N_d(n)$ points in general position in $\mathbb{R}^d$ there is the vertex set of a cyclic d-polytope with n vertices.*

Note that in the above theorem we cannot replace the cyclic polytopes with any class of polytopes of different combinatorial kind: one may select any number of points on the moment curve yet every n-element subset will determine a cyclic polytope.

Next, suppose that d is odd. In general, any cyclic polytope with $d+2$ vertices admits a triangulation into $(d+1)/2$ simplices, and also a different one into $(d+3)/2$ simplices. Thus, Theorems 13 and 14 have counterparts in every odd dimension [50].

These arguments however cannot be extended to even dimensions: it is known [42] that every triangulation of a cyclic d-polytope, d even, consists of the same number of simplices.

8 Further Problems

Let $h(n,k)$ denote the smallest number such that among at least $h(n,k)$ points in general position in the plane there is always the vertex set of a convex n-gon such that the number of points in its interior is at most k. Horton's result says that $h(n,0)$ does not exist for $n \geq 7$. In general, Nyklová [35], based on Horton's construction, established that $h(n,k)$ does not exist for $k \leq c \cdot 2^{n/4}$. She also determined that $h(6,5)=19$, yet another step towards the solution of Problem 6.

The following problem was motivated in [30]. For integers $n \geq k \geq 3$, let $g(k,n)$ be the smallest number with the property that among any $g(k,n)$ points in general position in the plane, there exist n points whose convex hull has at least k vertices. Clearly $g(k,n)$ exists and satisfies $f(k) \leq g(k,n) \leq f(n)$. Based on the results of Section 2 one can easily conclude that $g(k,n) < c_1 n + c_2$, where the constants c_1, c_2 (dependent only on k) are exponentially large in k. The true order of magnitude of $g(k,n)$ was found by Károlyi and Tóth [31]. It is not difficult to see that $g(4,n) = \lceil 3n/2 \rceil - 1$. In general the following bounds are known.

Theorem 16 ([31]). *For arbitrary integers $n \geq k \geq 3$,*

$$\frac{(k-1)(n-1)}{2} + 2^{k/2-4} \leq g(k,n) \leq 2kn + 2^{8k} .$$

To obtain the upper bound, peel off convex layers from a set P of at least $2kn + 2^{8k}$ points as follows. Let $P_1 = P$ and Q_1 the vertex set of its convex hull. Having P_i, Q_i already defined, set $P_{i+1} = P_i \setminus Q_i$ and let Q_{i+1} be the set of vertices of the convex hull of P_{i+1}. If there is an integer $i \leq 2n$ such that $|Q_i| \geq k$, then we are ready. Otherwise we have $2n$ convex layers $Q_1, Q_2, \ldots, Q_{2n}$, and at least 4^{4k} further points inside Q_{2n}. Thus, by Theorem 1, P_{2n+1} contains the vertex set of a convex $4k$-gon C, and the desired configuration of n points whose convex hull has at least k vertices can be selected from the nested arrangement of the convex sets $Q_1, Q_2, \ldots, Q_{2n}, C$.

Open Problem 12. *Is it true that* $g(5, n) = 2n - 1$*?*

Open Problem 13. *Is it true for any fixed value of* k *that*

$$\lim_{n\to\infty} \frac{g(k,n)}{n} = \frac{k-1}{2} \text{ ?}$$

An *interior* point of a finite point set is any point of the set that is not on the boundary of the convex hull of the set. For any integer $k \geq 1$, let $g(k)$ be the smallest number such that every set of points P in general position in the plane, which contains at least $g(k)$ interior points has a subset whose convex hull contains exactly k points of P in its interior. Avis, Hosono, and Urabe [2] determined that $g(1) = 1$, $g(2) = 4$ and $g(3) \geq 8$. It is not known if $g(k)$ exists for $k \geq 3$. It was pointed out by Pach (see [2]) that if P contains at least k interior points, then it has a subset such that the number of interior points of P inside its convex hull is between k and $\lfloor 3k/2 \rfloor$. A similar problem was studied also in [3].

Open Problem 14. *Prove or disprove that every point set in general position in the plane with sufficiently many interior points contains a subset in convex position with exactly 3 interior points.*

A first step towards the solution may be the following result of Hosono, Károlyi, and Urabe [25]. Let $g_\Delta(k)$ be the smallest number such that every set of points P in general position in the plane whose convex hull is a triangle which contains at least $g(k)$ interior points also has a subset whose convex hull contains exactly k points of P in its interior.

Theorem 17. *If* $g_\Delta(k)$ *is finite then so is* $g(k)$*.*

The proof is based on a result of Valtr [50] which extends Theorem 9.

Note Added in Proof. The answer to Open Problem 2 is yes and the proof is quite simple. Open Problem 5 was solved very recently by Pór and Valtr: the answer is again yes, but the proof is not that simple.

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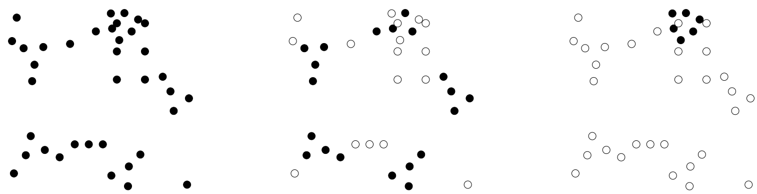

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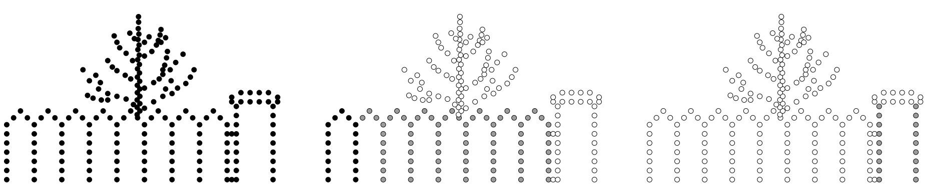

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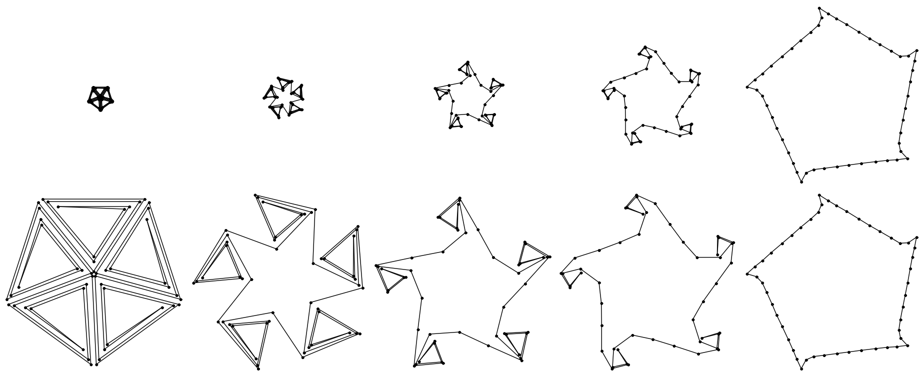

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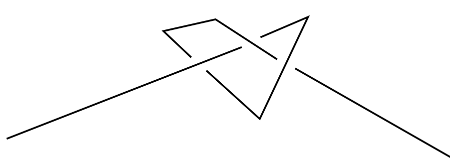
2 h n] h

n a n n a n a a y a n]
H a h h a a a n a hm n a ana y
n h h h m n n n n h an m y
n y h y n n a fin n m h a h m fi a n
h n a an y y a y a] h m an
ha n a m ; G] h h y
Un na y a hm m y a y m fi a n an
a a y many m n a a an G]
n y a hm ha n y ana y n h h n a n a n m
m m ha n G] n a y
a y na a y n ha a m h na n h n
an a h n n fi]; n n y h a
n h 2] a h a n a y na a an
n n y n n n a ay a h n n y
]; 2 an am a a n O h
m a n n h n h
n G] an a n H G]



A l l l 5 b s C 98, 99

na y ana h n n n n n m n n h h han
y na a an a h n an y na y an n fi n
an h h m n n O] n y h h a h n m
m any m h h h han h m n na y
h a m y any a n n a n h
m n n h h h h m n n h a n
h h am h an y n nn a h
n m n n h h han 2

a

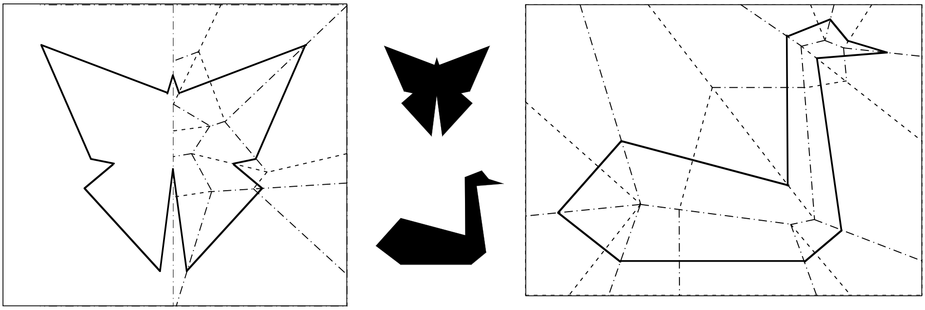
a n am ha a a n n ma h ma a an m
a na n h a fi n y a a n ma y
a y n h a a a an an y any n n
m n ha h an n h a an n y
n n ma y a ann a h may h
n y ma y a n a n n m m m n
h a n ℝ H h h m m n a
a m h n y n
a a a *t f* *n* h a a n h an an
h a m n a y h n y n y h n n
m n a n h h fina a h a ; n h a a a
n h a a a a *t r m*

m h n n n am ma h ma H a]
h *r* *tt rn* ha m n n a a am ha h
a h n h a ha a a am a n
ha a a n ha a n a y an ffi n n n a
n n H a] h h a yn m a ha a
a n n an Hay H] ha h n ha h n m
ha h y ffi y h n n n n y m y
n fin n an a a ha a n n n h
a h n n a n a n a n h n a a am
an n n a m H]

m n n a n H] *m t t n r m* h
a hm a a n h a a n y an an]
h ha h n h n a a am a m h h a
am m a n h a y a m a na m y h
n ha n a m a na am ; y y h
n h ma n h n a]

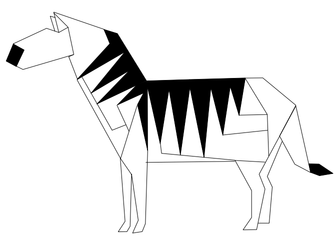
On n h *f* *n* *t* *r b m* n a h a
a ma n m a h an n h ha ha an
a h n y an a an h an h n ma any
an a h ana a h m h a h
m am n h m ha n ma n
ma n an] n a n a n h a h n
a h n n y a a n ma n n an Hay H]
a n n a n a n

n h n n a y n a n a n
h an a n yh a G n a y n n h an y h
h an a ffi n y a a n ha h h
n m h h am n an a fi ma y a
n H] h a ha a n n a h an h y n
a n n n y n a n an h a
n ha ha h h a a h n a h a a n
n a y h ha n a a h a n a n



C s s l l s l s
s b l s ll s s
l s, s s s

n n n y n n a a yh n an h a a a a ay
h y a y a ffi n y a a y ma h n a
2 a n ma n ma n an h] ha h n ha h
an a h n a y an h a hm
n h m a m n n n h ffi n y h n
ma n n



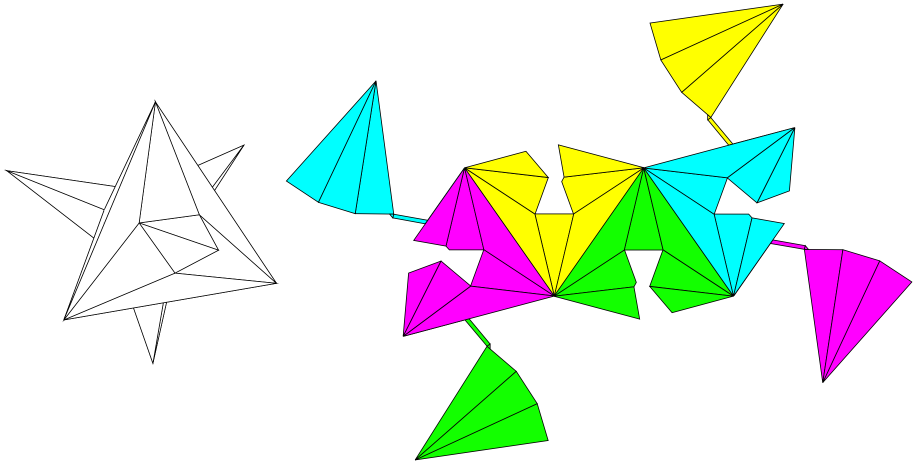
A l s , bl s
s , s b ll 9 , 9 – 0

n n h m n n a a a a n an n
n a a *m f n* y a *m* a an h
h n a an a a a h ma h m n a n a y h
ma n n ma y n y m h m na a Ga
n] a m n na mm n a n a
a n n a n ma ha a a a an 2
an a n ma a y a n m a h n a n
n n h m y h fi m ma n n; an ha n
an n n n h n n H] n y n n
ma n ma n h h a an na] h m
y h n m h a n n h m m

h a an n ay a a m h n a y an
n n a m ; a ay m a n a m
m h n a y an n n a n a m
n y h ma n n h m a na n a a y
h am n h a a h a n n an a
a a y m]

oly a

Un h h m h a a n m n ha a
n h n yh a
a n m h h h a y n yh
n an a n m an n n h an h
a h O] h n n a] an ha m
an a a a a n n man a n h a h m a n n
y n ha h an h n y a a m
a a n ha a a m n y h n h O]
ha a an m n n a an m y a h a y
h n h n a a n n n a n n n
a yh a
n a an n h ﬃ n y an am n ha
n an n a n a a fin a yh n *t*
n n a h h n a n yh n
y a y n yh n ha h an n n n
ma n n an an n y n] ha h n ha
h an n h a yh n h m m h a h an h
y a a an ha ha n n n n a n n n
h n n h m y n h h a n a y
n yh n an n ma n n
n h n n n m n h a a h h y yh n
h m m h a h ha *m* n n n n n a y n
a n n n ha y n yh n ha an n n
n h m a n a h a h y O 2]
many n n n yh a a ha h n n am
a n h yh n a ma n ma n
O ma O n an h] ha h n
h n many h na yh a n h h an n
y a h h ma n n h h a h na yh a an n
h n y n n na ha n n n a h] a y
h n h a n n a a m han ° ma a
h h yh n ha n a y n n n y n
a
n a n n n yh a n m ana y n an
n h m n y n n yh a an
O O] ha h n h n yn m a m h h a y n

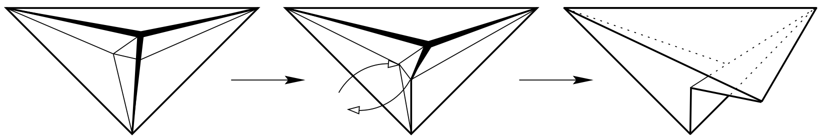


L l l l l A l
s ll ss s

ha an n ha an n a n yh n an
h a h n n n n a m h n n a
m n a y a m am ha ha many n a
h n h n O a O] h h a
h n m a an a a n an n n a
a n n ffi n a hm n h n an n m a
n n n n n a n O] n n n n m
ma n a y n a h m an m ha any n
n an n a n n yh n h ffi n y an
h yh n n
n n yh n n m m n n h an
m m h n n h h m n n G n any yh a
m an ℝ h h ℝ a n ℝ ha h a h
yh a m ma a mm n an an n h n ma ha
an h h a a y m n n ffi ma n h
m n h a h m a a n
a n *tt n* h a a yh n n a a a h
n h n h a ma n ma n an] ha
h n ha n yh a an h na yh a an a n am n
h a n am h n n n h ha y
yh a m an a n

5 on lu on

h a a n an n n many a ma h ma a an m
a na m h ha n ma n y n h many
m n a an many m m an m ma n n



l , l s
l

am m a n n yh a ma n n n n h
na m n h a a n n n yh a a y
ha a n a a ha n n a n y
n h h y ha n h h a n a
m ha n h a a h a ha ma n n
am n n a ma n a a a m an n
y ha h h many n n m m h
nn n n a n m n a h a n n an
n n ha ha n y n n n

no l g n

han my a h h h m ha n y a a n n h a a
n an n n

n

A C 00 H A , s , C , H ss , s,
l l *e* *c* , 0 57– , 000

A 00 A , A , , L ,
ll, , W l ? C
s s s C 00 0 , v 000 `h ://` `o g/`
`/` `/001102`

AC 00 A l , C C s, , v , s , H
, v s, l , s , ss l
l s *P c.* . *c e e* . *e* ., L
s C , , , v 000 *c e e*
e e

A 0 A l , , s , H , v s,
A ss, ss v l s .
e . *e* ., 00

A 99 A v, , ll C v l l
s ℝ s , 999 `h ://` `h` `/` `/`
`o` `/ o` `g o go 10 27`

A 9 A v v l s l *c e e*
. *e* ., 8 9– 50, 99

98 l, , , A L b , v s, ,
bb s, W s s l s l ss s l
l *P c. 10* . . *e* ., l,
C , A 998 h :// g g / g / o g /
g o g gz
99 l, , , L , A L b , ,
v s, bb s, , ss , W s s
L l l l s l 0 0,
C ll , 999 A l v s *P c. 10*
. *c e e g* , l , l , 999, s
8 –8 7
0 l, , , L , A L b , ,
bb s, , ss , W s s A
l s s l *c e e* . ., 00

0 , , s , , A l ,
l bl l v s . *e* . *e* .,
00
H98 , , s , H s A s l
P c. e . . *g* ,
s l lb , l , 998
L 99 C l, , L , bb s, A ss
C v l s *P c. e* . . *g*
, v l 7 *ec e N e* . *c* ., s
5– , C , , 999
H9 H s l *P c.*
. *c e e g* , s 75– 8 , A l , 99
C 00 C ll , , l l s
v l l l s *P c. 1 EEE* . .
. *c* ., s – , , C l , v 000
C 98 C ll H s v l b s l l
v ls s s . *e c* , 7 8 0 7–
0 9, 998
C 0 A C lv , , , ss, ss C v
l s s l s . *P ce* . *e* ., 00
C 99 C l l s l *P c.*
11 . . *e* ., v , C , A 999
h :// / o / / o / 17 gz
0 L s l s l
P c. e . *ee g g c e ce*
E c , , C l , 00
L98 , L , A L b l
A , , b , s, *e e P e e*
. *c e e* . *e* ., v l 7 *ec e N e*
. *c* ., s 0 – 7, , , 998
L00 , L , A L b l l
s , 000
L 00 , , A L b , l s,
l s, s l s l s l s b
l s l s l 0 9, C ll ,
, A, l 000

L 00b , L , A L b ,
l s l s b l s l s *P c.*
. *c e e* . *e* ., L s C , ,
, v 000
00 , L , ll l s l
s l l s s l s l
. *e* . *e* ., – , 000
77 A *e P e'* *e e e e*
e e e e e ec
¨e e U e e e
ge e P e e e V Ab s s, , ,
977 l s sl *U e e g e e g e Z e*
c c e e E e e e , 5 5
L 98 H v , L , bb s, H , W s s C
v s s l s *P c. 10* .
. *e* ., l, C , A 998 h :// g g /
g / o g / g o g gz
5 s bl 7 *e* . . , 7, 9 5
H 0 v s, A H , A s , ss, ss l
l s b *e ¨ge ge e* ., 00

8 b s l *ee e*
e e c e e , 7, s 0–7 W H
C , 98
95 b H v l *e c* , 5 – 0, l
995
H l9 H ll s s *g . N e* ., 00 5– ,
99
s9 s s l , ,
P c. e . ee g g c e ce c e c g ,
s 5– 9, s , , v b – b 99
89 s l b s s v ll s s
H H , , *P c. 1 e . ee g*
g c e ce ec g , s 9– 7, , l , 989
A b s v s *e ege ec g*
e , 7 5 – 57, 990
95 v lls l s l s
l l . *e e e* ., – , 995
L 9 L A l l s *P c. 1*
. . *e* ., s 98– 05, l l , A, 99
L 9 A L b W l l l ?
l 0 8, C ll , 99
L 7 W L l s l l *e e* ,
75–80, b 97
LW95 W L H W s s l s l l s
l s *c e e . e* ., – 0, 995
87 ll, , C H s
s bl . ., 7– 8, A 987
9 ll *c g* v bl s, 99
9 l bl 7 *e* . . , 7 – 77,
9 9

98 l l l *e e*
P e e . c e e . e ., v l 7
ec e N e . c ., s 58– , , , 998
l7 ll s s v s *e . e c* , – 5,
97
89 C v *g e e c P e* s s, s
H s v s , 989
75 C C v l s v s *. P c.*
ge P . c., 78 89– 0 , 975
00 A b l l ll b
l *P c. 1 EEE . . . c* ., s
– 5 , , C l , v 000
99 ss C l l l l *P*
cee g e V E c e e e e c , C s ll ,
, l 999
99b ss s s s *P c. 11*
. . *e* ., v , C , A 999 h :
// / o / / o / 1 gz
0 ss A l ss s s *e ¨ge ge*
e ., 00
W 9 W s s Al ss s l l v
e , –50, 99

h h M c P p

An ine De ' K ei D i rii se ni n s n ri

[1] s S a s cal a a cs r c a r l k a a
de a m ac
[2] E r c s r O ra s s arc r c Sw z rla
da ma e c
l rs c l O ra s s arc W rla s
d a ec n de n
[4] k s c l a a Sc k a a
[5] E ESS r al s a ´ a q S c al s ar s ra c
de a e e

bs ac W c s r c l ra w a l ca s w ll k w c a r al za r l s r c l n a s r la s r n scr r c l s as as n as a s 44 r c s ar r s lar s r sl k w s a c as 7 84 r c s l 1 r s s s lar s r r w ra r w s 1 8 r c s 8 s al r c l scr c s s s r s a s c c r c l r w s c c a a ac c r la s ar als sk l c l lar r al s ral a r s r c l n s W l r c c c s a ars a a r s sk l n w c c r a c r c s r a c s c a r al a c a al a l ca s s c c r ar s ar c lar a r s c sk s rac s r s

O O

C in ri p pes i e p pes rising r in ri p i i i n pr e s re en ri i r e er firs ses n en s en e s e in ri e p si n rs e en r s ins n es i e ese p pes rn e i in r e r en er i n g ri esigne r gener p pes i r e g ri s sing eir ri in ri e res n e i i s rprising s r ng per r n es r e p e S F R p e rge ins n es e *r l s l s p ly p* e *l r rd r p ly p* n e *u p ly p* e p i ing eir s e r gr ps n si i r ein in i i n i s s e r gr p e se i s in ri s r re r i ise en er e e er i es n er in ri p pe: *h r p ly p* e firs re si efini i ns n presen s e ppi i ns e n n in ri p i i i n pr e s

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 125-136, 2001.

e ( ) i ensi n p pe c is s in r e s e n e e in i en e e rs e s re pre ise gi en s se S e u e er ine S nsis s e p irs e e en s s e ne is in S δS e en e e n i s in i en e e r in ; is δS i e ne is in S n er ise r $\leq$ $\leq$ se n i n e se e er r e i se n i s in i en e e r s δS re nsi ere s r in es p in in e p pe c is e n e s n e u C is e ni n n er s e p pe n ne i s re i n e e ri p pe n s e efine in er s fini e e ri sp e in e ing r se s e nsi er e ing ine i ies:

$$x \quad x \quad x \leq$$

$$x \quad x \quad x \leq$$

in e e ( ) e s i efine e r en n ing e er e in e e ri p pe e ( ) resp ( ) e s efine resp n e seen s ri ng e resp peri e er ine i ies r is n e x n i e e ne is e ni p ns n ip e e e re e r s e e ri ne e p pe c is e n e e er i es e e ri p pe e in i fini e e ri sp es is e ing: ere is n r rresp n en e e een e e e en s e e ri ne n e se i e ri s n p in s n e e e en s e ne rresp n pre ise e se i e ri s n p in s re is e ri e e e in s e is e s e s ini is s er r e ( )

ne e i i ns r e s ese p e r es r eir p p i i ns in in ri p i i i n e s i p r n eing e n i i pr e s i en gr p n n nneg i e eig s $_e$ e ssigne i s e ges e u pr e nsis s in fin ing δS se eig $\sum_e$) $_e$ is s rge s p ssi e is e n n N p e e pr e se ing $_e$ i e is n n e ge e n n si er i ss gener i e p e e gr p n en e pr e n e s e s ine r pr gr ing pr e er e p pe c s s: x s e x c in e e e ri p pe is re i n e p pe p i i ing x er c ins e pr i es n pper n r e pr e C nsi er n e p e e gr p ; n ins n e e ul d y w pr e is gi en n nneg i e e rs in e e : p i c e n re ire en e r e e e e : e en es e s se sp nne e e ges in en e s e gr p en es e s r e e ge e s in e s pp r e see e ni s e een s n in e p e e gr p e s s ng n e ge e s n e ee c e s se s e is s e c s l A ne ess r n s ien n i i n r e si i i is: p ir c is e si e i n n i c x is i er e e ri ne see r e p e e ri ng e e in e n e seen s n e e en r s e pr e i *c* n *c e* *e* er ise s rresp n s *c* *x* r *x* in e e ri ne ere re e e ri ne is e ne e ne e si e i i pr e s r e i e s se p pes n eir pp i i ns in in ri p i i i n e re er L n

O O Y O

in t ri nd tric r rti

e p pe *c* is ( ) i ensi n p e r n i er i es n is p pe e s e i ensi n i ( ) e s ins ri e in e e e e *c* i e i n r $\leq$ is e s see e p in ω – – – is e en er gr i *c* n n is s e en er e sp ere r i s $-\sqrt{}$ ere e s ie An e e e ri p pe n ins e e p pe n e er i es e p pe re er i es e e ri p pe n e s re pre ise e in egr er i es e e ri p pe e e ri p pe r ps e p pe *c* er ig n ee in i i n e er i es e ges n es *c* re s es r es i is se r An s re en n *c* n n ; in er r s is *u s* *r l*; is e s e e n e n e i s in egr er i es i e e s e e n *c* is n in e s gr p e s e e n e e ri p pe i se e re e s e e n p pe is e gr p r e i s er i es n e ges i e e i e ers e p pe n e e ri p pe s is δ *c* n δ e i e ers eir re n e re e δ *c* n δ

ne i p r n e re e e ri n p pes is eir er rge s e r gr p e re e *sy* *ry r up s* p pe is e gr p is e ries preser ing re pre ise r *s* *s c* n re in e per i ns n n *sw h r s y u* n r e e *s* i en δ *S* e s i ing re e i n) is efine) *x* ere *x* i δ *S* n *x* er ise As ese s e ries preser e e en re i ns n e ine r in epen en es re p r i i ne in r i s es e i en n er per i ns n s i ings

rtic t M tric yt

e re s e res s n e er i es e e ri p pe n e L *d l u ur* e s re e n in egr er i es A er er i es i re n r i n re s e *r l s s* er e C nsi er e ing ppings

$$v \quad \phi \; v \qquad\qquad v \quad \phi \; v$$

$$\phi \; v \quad v \qquad \phi \; v \quad v \qquad r \; \leq \quad \leq$$
$$\phi \; v_{,} \quad v_{,} \qquad \phi \; v_{,} \quad v_{,} \qquad r \; \leq \; \leq$$
$$\phi \; v_{,} \qquad \phi \; v_{,}$$

e er i es ϕ v n ϕ v re e ri i e ensi ns v e ϕ v {) ϕ v n er r s e ne er i es re e r i n nes e – – e r i n er i es re e s ie n in e e n i r i r e e *u s δ S* – –*δ S* is nne e gr p e en e i s p e ri ere is e eng s r es p r in r en *x* : is e e *r* *r* n e se *x* $\overline{\tau\;)}$ An er e e r *x* r s e gr p is e *r ph* er e see ig r e gr p s gr p i – – e er i es 8 e r n nne e gr p e e ≤ n n – – e er e is p s i ing gr p i e e *d* *I d* en es e n er e s n ining e er e *v* n e *d* *y I d* en es e n er er i es en *v* i e r ing n e ge i *v* e ing is s r ig r r pr e

r iti n *h r s h r p ly p r p r d r s s sy ry r up L v r s d* A *s d y h r r d y h s v d v h l r pr s h* A A *d*

in e *c* n *c* e er i es n re e n s r ing r i e er i es re 6 s n 6 n i s i e r r i s e e ri p pe s er i es see p r i i ne in r i s: s n i s n r i ri i e ensi ns; n s er i es see p r i i ne in r i s: s n i s r i s ri i e ensi ns – – e r i s n er r i n r i s ee e ere e n ni represen i e er i es e e ri p pe n n es re gi en i eir in i en e n en

r p r y 1 e *v* e er e n *δ S* n en ne s: *I d* ≤ *I d*) ( ) i e i n r *v* *δ S* re er i *v* is ri i e en si n *I d* ≤ ( ) n i *v* is r i n *I d* ≤ *I d*⁻) ( ) i e i n r *v* *δ S*

r per is i s r e in e ere e r i s re r ere e re sing es e in i en e *I d* e firs r i $_1$ is e r i n r i n r i s re er e n i r i e ri i e ensi n r i s re r n

Tab 1 1 r s r c s

Or ˜	a cal r r s a r	˜	*j*˜	˜
˜1	)	1		4
˜	$\frac{2}{}$ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)	49	49	4
˜	$\frac{2}{}$ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)	4	94	7
˜	$\frac{2}{}$ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)	4	7	1 44
˜	$\frac{2}{}$ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)		89	4
˜	$\frac{1}{4}$ 1 1 1 1 1 1 1 1 1)		9	1
˜7	$\frac{2}{5}$ 1 1 1 1 1 1 1 1 1 1 1 1 1 1)	8	7	4
˜	$\frac{1}{}$ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)		7	4 48
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C n e re n er ines e e re e nne i i e s e e s r i e in e n eri p pes ere re i C n e re s e s e in n i e in e s e e n i p ing i s i e er s is δ $\leq$

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As s e in r p si i n e *h rh d* is e se er i es en gi en er e is e i en p per i ns n s i ings r er i es e nging e s e r i is pr per e s e ing *r w s u r l r h* e in s r ines re e p i n e n ni represen i e v e r i gener e er e v n e en er i n e neig r N e er e v r ing r n ini i er e $v_{s\ a}$ e g ri p es e n ni represen i e $v_{s\ a}$ en er es i s neig r $N_{ta\ t}$ i en ifies ne r i s n ine in $N_{ta\ t}$ p es e is n ni represen i es n en pi s p e ne n ni represen i e in se neig r is n e p e e g ri er in es en ere is n re s n ni represen i e in n p s in e e s e e n p pe is nne e is g ri fin s r i s *r ws u r Al r h*

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e r i ise en er i n g ri per r s ssi er e en er i ns r s ers p pes ne r e r i neig r s ins e per r ing ne rge ssi er e en er i n e e p pe e p i n is in epen en e i e e ini i er e $v_{s\ a}$ A ng e n n er i es ; n e s i e r $v_{s\ a}$ is e n i δ –

n se er ig egener e s r ine p ing e n ni represen i e s e e rge n er i es n s e e neig r s ig represen rge r i n e e p pe is e se r e neig r N $_)$ s $I\ d$ $_)$ ( )

r e g ri ge s e r i ise in i en e $I\ d$ si p e ing i ine i is s isfie i e i s inp r e neig r en er i n s r ine n pr es e en Ad s p is s r ine n ing e n er i es er e e i en e n ni represen i e v is n in N e ge e *r ws d y l* is e ri Ad i Ad $_,$ Ad $_,$ e n er er i es e r i en v e r i s re r ere firs e re sing es e in i en e $I\ d$ en e re sing en Ad n en in re sing r i si e e s ss e e n e si e ne r i ; r e p e e

~) en r e ri Ad e n s ge e si e e er r i s sing e ing e s re i n: Ad , Ad , ee r e p e e ere e r i ise en e is gi en r e e ri p pe n n es e firs r e is s r i ise N_1; is e 6 neig rs er e e nging $_1$ is r e p e Ad_{1}, in e r n e ns is en er i es e nging e r i in e e e s in i en e rigin δ re pre ise e ( ) ri ng e es s n e re e r e e ri ne rresp n s e er e en v δ n er r s e en Ad) Ad_1 e s e n er e re e r s e e ri ne e re e r i s n er per i ns e e re e r s ere n s 6

Tab Or w s a ac c a l sk l

	$\tilde{}_1$	$\tilde{}$	$\tilde{}$	$\tilde{}$	$\tilde{}$	$\tilde{}$	$\tilde{}_7$	$\tilde{}$	$\tilde{}_9$	$\tilde{}_1$	$\tilde{}_{11}$	$\tilde{}_1$	$\tilde{}_1$	$j\tilde{}$
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$\tilde{}$	4	7	1									18		7
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$\tilde{}_1$	8		4					1					1	4
$\tilde{}_1$	14	1	1	1		4	4		4		4			4

r e p is e is n ni represen i es v r is e re e p Ap r r er e en er i n e g ri p es e r i s in ri n s Ad I d n e r i ise en e Ad re e s e s e e n e n er er i es is si p $\sum$ n e is er i es n e gener e e i n e s e r gr p n e represen i e v

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e e ris i s presen e in is se i n re i r er in ri p pes r n enien e e res ri rse es e e ri p pe nser i n g ri s s n e ig egener e er n pi ing g ri s see r e i e presen i n e in er e en er i n e s e e ri p pe is i e egener e e in i en e Id) ( ) is rger n e i ensi n ( ) s e se n inser i n g ri r e neig r en er i n s r ine: e c i p e en i n e e es rip i n e n e re in er e s ss e e neig r en er i n s r ine is per r e n inser i n g ri e e r in i es e en e neig r s ig egener e p pes ig ie e n e r nge pr e s rren s e inser i n g ri s n e e presen e ris i s ressing is iss e

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r e e L C n e re gi e e ing p i n i p i i n: e en er i n e e re e r s gi es e r i s e in e e n er e re e r s e e ri ne Ad) ig e rge r i n e n er er i es e e ri p pe e p i n g in e i i e ere re e pr p se u s ur i n e seen s p e en r e L n e re gr p i n p i n

C ur r 6 e res ri i n e s e e n e e ri p pe e n n er i es is nne e

r n p ir er i es i e C n e re i p ies ere is p e s ining e C n e re e ns ere is p e n n s er i es ining e n er r s e er i es r d s n u s in e s e e n n e er n i e C n e re e ns e en er i n e e ri ne is en g in e e ri p pe ; C n e re e ns e n in i en er ing N) see e e r r i r r gr p s ese re e r in epen en re s r ng e ie e e r e n r ≤

H ri tic i in Hi n r cy

C n e re s r i s n e n e ing e ri ne s ip ping e ris i : isreg r v i v δS n er r s isreg r e neig r e s is essen i e e ri ne is neig r is e ie e e rge rgin e rges s e e pe Ad) Ad) see r per n e e r n er r s e e ris i re es e r es neig r en er i n e s re e s re gni e s δS is ni e r eri e i s in i en e: Id) ( ) ere re isreg r ing e e ri ne nsis s si p in sing n n s ini i er e $v_{s\,a}$ n i ing e in p e r i ise en er i n g ri in e ing :

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r e e ri n n es i e *Ad*) 6 i e e n er er i es 8 e en er i n e er neig r s gener es i ip i i $\sum$) *Ad* er i es i e ess n e n er er i es ne n e si ge e neig r e r e r i ise en e *Ad* ing e n n e r rresp n ing e s s e e: *Ad*), *Ad* ,) ere *Ad* ,) is e n er s en *v*

r iti n *I ru C ur w uld r h h “ r C pp H ur s s pl d s r p h r p ly p y r ly ry s ll r h r s*

ne n r er e re se e p i n i e s ipping n n e r i i e ig es in i en e e s r i s i r i r ri se in n e pper n *I d* *a* n e in i en e ipping ig egener nsis s si p in e ing ifi i n e in p e r i ise en er i n g ri :

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n is se er ifi e r p e e es rip i n is e res ri i n e s e e n *ta t* n e in i en e r i s $I\ d_v \leq I\ d\ _a$ is n ne e is e ris i is p r i r s i e r p r i en er i n p rp se n e i e e ini i er e $v_{s\ a}$ e e ri i r see e r p si i n r e e ri p pe n n es e n e $v_{s\ a}$ *v* ; is e ri i e ensi n i e r ig es in i en e *I d* e e pe e is pe er e e nne e n r i s n in ee e neig r *N* n ins represen i es ifferen r i s An er i e is $v_{s\ a}$ *v* i *I d* 1 n *Ad* 1 e neig r p i n s r ine s res ri e er i es s is ing *I d* ≤ –( ) i e r e i ensi n ( ) e n i in i en e *I d*-) ( ) esi es *N* 1 e g ri p e neig r s i in i en es gener ing $\sum$ $I\ d_v \leq$ *Ad* 6 er i es i e ess n e n er er i es i is e ris i ppr pr e e en g s is in n er er i es n ins represen i es r i s An er re r e e re is sin e e neig r s re gener e e e s e e en ess n e s e neig r en er i n s r ine is per r ing e re e e i i r e pre i s *r s pp h ur s* se issing en ries e e *Ad* i e e r s rresp n ing e r i s i ig in i en e *I d I d a* n e p e sing e re i ns: *Ad* , *Ad* , n p r i r e n firs p e *N*-); is n n er es *Ad* -), n sing e -) ge rresp n ing n en se e i er i e in s e e re ining n n n

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sing e e ri ne s ipping e ris i presen e in e e en er e r i s 8 e is n ni represen i e i eir en n in i en e n espe i e en e *Ad* eing rge e in e in is p per e re er `tt www t t a a a t` ere e i e presen i n is i e r e p e e n i r *Ad* s n n n er en ries *Ad* -), A s r es rip i n is gi en in r p si i n

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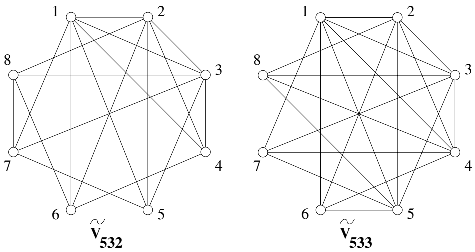

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e presen e n r i ise en er i n g ri r in ri p pes i rges e r gr p n p r i r e p e 6 er i es ig egener e i ensi n p pe efine i s e s: e e ri p pe n n es e es rip i n nsis s n n ni represen i es n e n e re i is p e e e r i ise in i en e en n s e e n re s gi en i e e e re e nne i i e s L n e re e r in ing se ppe rs e ne e in e res es e e n e n e re e er i es n r se in es e e n e in ri n p i n pp i i ns is n e re re s ie n p r i r e ris i s ipping e e ri ne is presen e e g ri n e p r e i e er e si n ine i e e ris i ig er i ensi n ins n es e e ri p pe n er in ri p e r er e en er i n pr e s e s e i e e rges pre i s p e e ri p pe s n r i s er i es $_8$ s e s r i s n ere re e rge en g re e s e gener e res e e ri p pe n n es n p r i r e s e e n $_8$ s gges s e r i ise en re i ns e een e s n i s e ri i e ensi ns n e r i n r i s: e r *Ad*), n e n *Ad* ,) s e n n n er en ries L in ing se n e re e n i s re in r i ise en e ri i e ensi ns n e r i n r i s re r i ise nne e ng e se es e ri i e ensi ns re e nne e ng e se es n n s e e r i n r i s s i e res ri i n e n n r i s is nne e n se n e re

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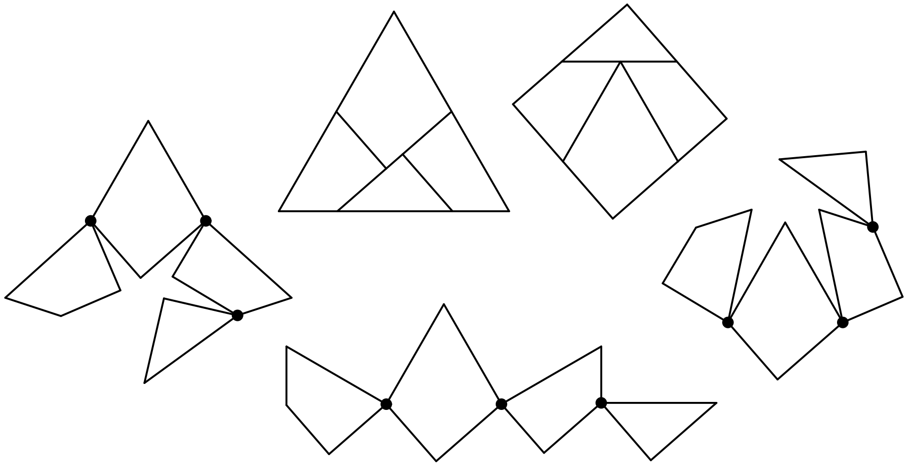

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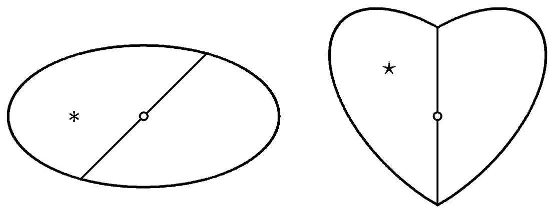

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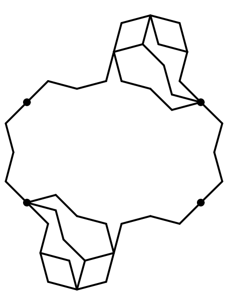

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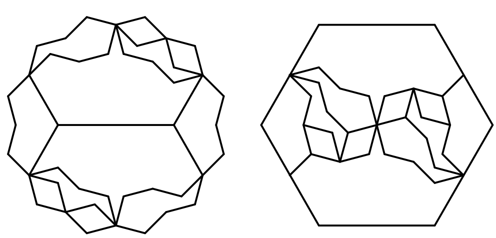

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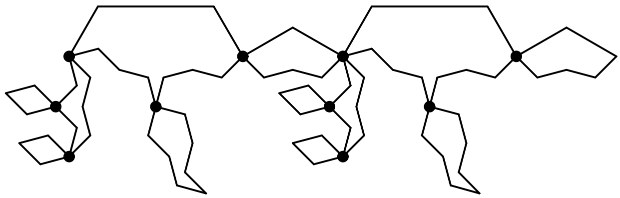

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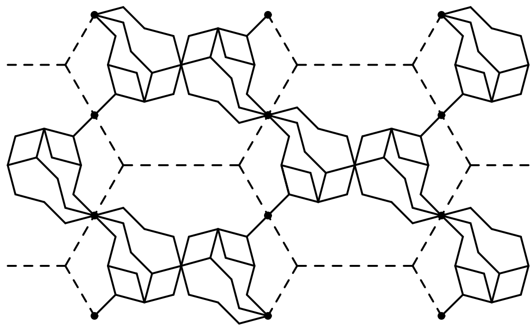

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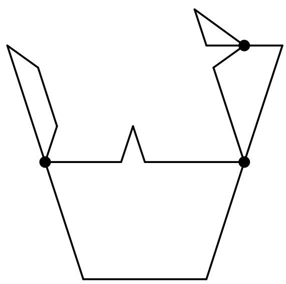

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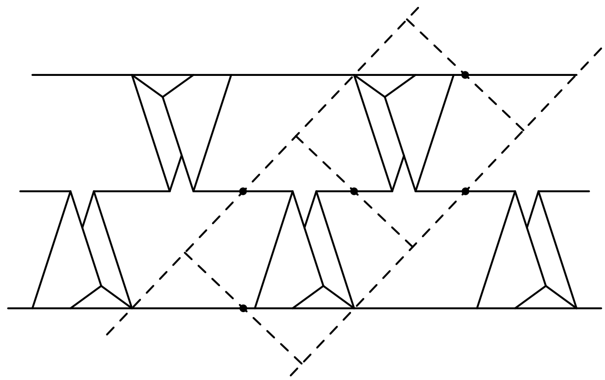

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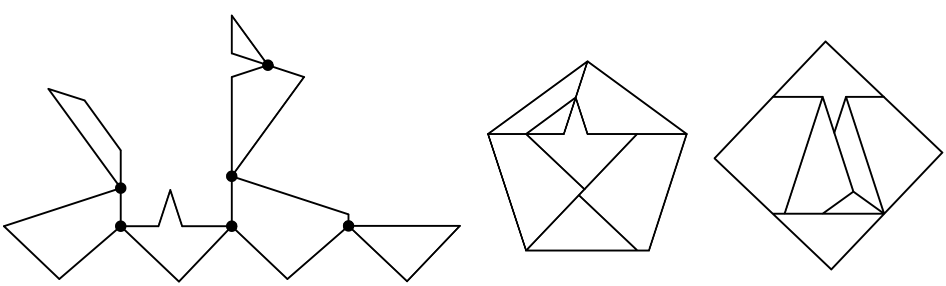

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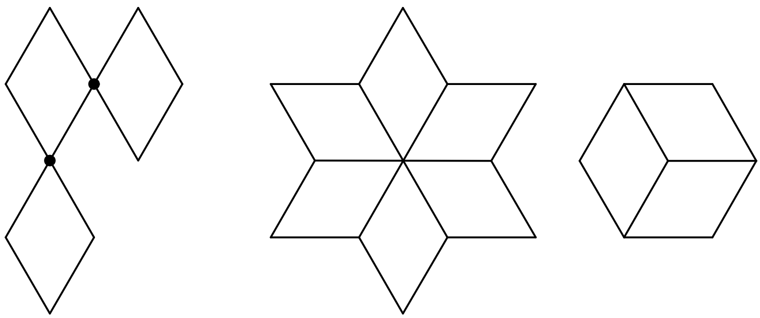

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o 3 *t* *g*- *g* *t* *t t*
t *g* *g* - *g*. *W* *t* *g g t*
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g .

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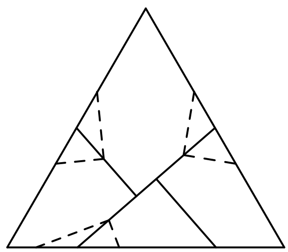

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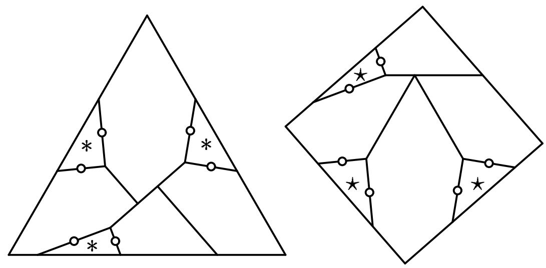

ig ri u r

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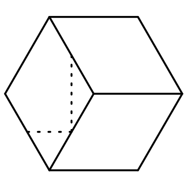
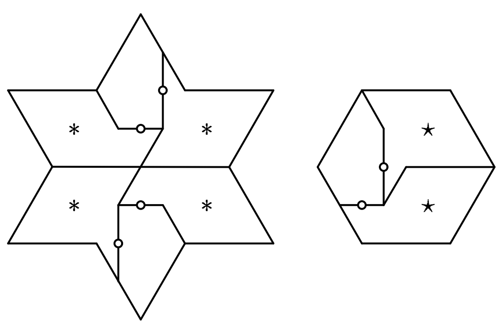
i w i .

o *t* *g* *t* *t* *g* - *g* *t* *t t*
g *m* *g* *t* *g* *g* *g* - *t* .
m *t* *t* *t* *g* *g* *t* *t* *t* *g* .

P . *I* F i i i pi
i pi i p i fl i i .



ig 3 A i c



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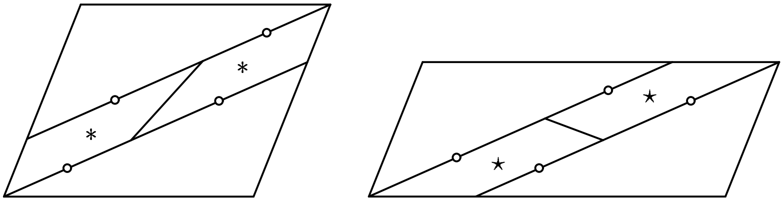

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o *P-t t t g m t*
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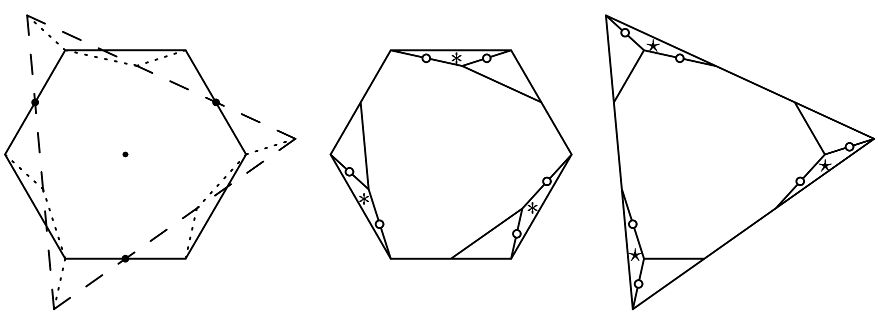

ig i i b i c i ri

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o *m t g t -t t g* (- *t t-*
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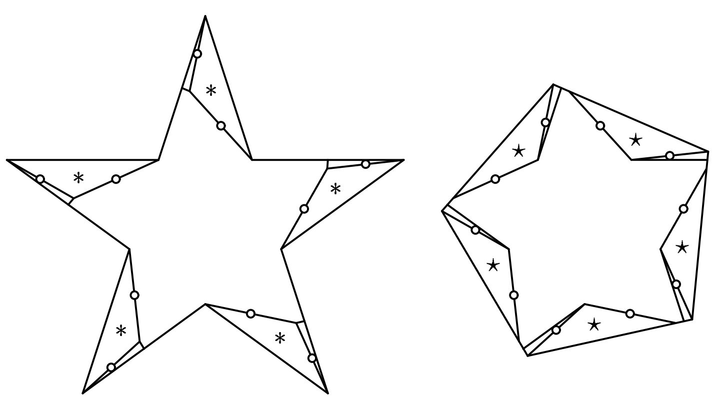

ig i i b i c i r

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t $3q -$ *t t g q q* .

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e e e es

1 i A i Gi u N ur u i c i i
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t apa C CDC L tu t C put S ,
v u 1 63, 1 –2 S ri r r , 2000
2 F r B i *ta u tut* i ii r ru r u
Si i, M r r i i, 1832
3 L Bru r *t al l a D t at* c ,
r , M i , 1 63
H M r u A *at at al l* r , 1 2
ri i , M r i L i , vi i , ric Fri Hi
i c i i i *P t t Ca a a*
C C putat al t , c uv r, 1
6 H r u u ri u i *l D pat* , A ri 1 ,
18 6– c 26, 1 03
H r r u *Ca t u Pu l a t Cu u P l*
H i , L , 1 0
8 i i L r, r i ur i bj c
r U S , 2,631, 1 8 Fi 1 83
H r v *Su t* , v u A B c , B , 1 63
10 Gr N Fr ric i c i S i i M u cri , 320 , Au u ,
2000
11 Gr N Fr ric *D t Pla & a* bri U iv r i r ,
N r , 1

12 M r i G r r *S t a at at al Pu l &* *D* Si Sc u r, N r , 1 61
13 G r i r c i u j r b i bi A v ic r i i Fi ur i i b S uc *u al u u a a t at at* *C ll u al* , 10 228–23 , 1833
1 Br Gru b u G S r *l a Patt* H Fr , N r , 1 8
1 A H r i r i c i , N r , 1 8 Fir b i i r j c ri *P l al D t*
16 i i u r i i r *a a t t al S t* *u* , 33 1– 3 , 186
1 G A r r *S l t C t a t S t* M r , bri , MA, 1 2
18 H Li r r b 1210 A i c i ir ui r ri S u i *a at at al t l* , 63 66 –668, 1 6
1 H Li r A u ri r i c i *u t al a at at a* , 16 6 – 6 , 1 60
20 H rr Li r *t D t* N r , ri c , N r , 1 6
21 S L M G ic u c u i Su i i *P la lp a* *u* , c b r 23, 18 8–1 01
22 r Lur r H r i i c i ic u i u , i i i i 80 r uc b B r, i G r , 1 8
23 H M c u i c i r c i i ur c i u *at at* , 2 3– 6, 1 22
2 Sc b r *at at al p u* M ic A ci i A ric , i , , 1 82
2 Hu S i u *at at al S ap t t* r U iv r i r , N r , 1 6
26 S r *P l at at* r U iv r i r , r , 1 8
2 H M r ric i c i *at at* , 3 81–101, 1 0
28 i i c , i r *l t t* B & Br u , i bur , i i i , 1831 Fir i b uc i , i u b ir
2 ric i i *C C C l p a at at* r , 1 8
30 vi *P u D t a Cu u a t t t* ui B , L , 1 1
31 b r *t al l a at at al S t a l* uc i ub i r , S L ui , 1

O s s s

o o o o D v d pp po [2] d M U be

p n s
T n s y 0 6 0
p n C p n n n n n
Q n s n s y n s n n 6 C

a . n pl n p n s n n l p s n s p n p n s n n x lls s n n lls s s pl p ly n n y p n s n b n y) n n n b lls n s p n *n*) b fin s x l) n s *n* p n s n s p p s *n*)/ *n*) *n*)/5

t o tio

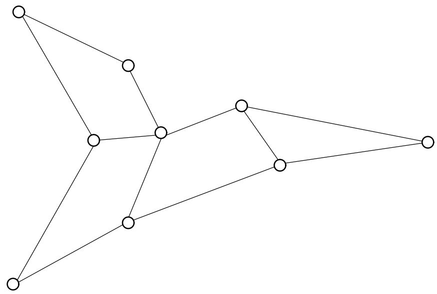
P o o po e o co ve b e b q o p oble d c e e eo e e do e e o po e pl e w c we w ll ll de o e b e po e ed o be e e l po o o ee po o l e b e o e e ve ce o co ve *k* co ve b e o w o po o ope e o c lled e l d p pe o do d ze e e o e v l e o e lle e e *k* c e o *k* po co co ve b e o ze *k* b eq e l l q e o ed b do o e v l e o e lle e e *k* c e o *k* po co e p co ve b e o ze *k* l e o *k* e ow o ll v l e o *k* e cep *k* I ow d 5 I ow 5 c F e ow 9 po e w o e p co ve pe o o o 5 ve co c o ow 7 o e e e e b l po w o e p co ve 7 o e v l e o o ow d e l z lo o d ope p oble o e e pe e l e l ow well l o o co p e p co ve b e we e ow po e w o e p e o ve o e co b o l e l o p o o po e wo d ee d e o e p e e ed 7 d

I p pe we co de e ollow v o o e co ve p o e e ve e o po we w o p o o e p co ve b e c e o o e b e o le ple pol o d eve po o e bo d o e e we c ll c e p co ve b e o *k* o *k* ve le *f* ep e e e be o cell

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 149-155, 2001.

ob ed c p o o e de o e e v l e o *f* ove ll e w po

Fo e ple 9 po e F e ve *f* Fo ple pol o w o de ce we c lw l e e o e v l ppe bo d o



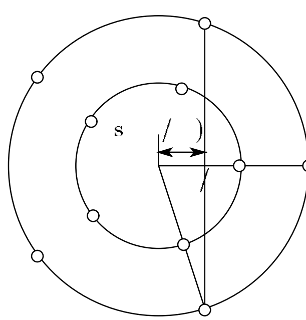
i . . T ns n p n s p n n lls n n ll

I e e ec o we p ove e ollow eo e

$$. \quad \overline{\quad\quad} \qquad \frac{\quad\quad}{5}$$

e owe o

w ow 5 e e e e w o e p co ve e o v e lowe bo d e ob e e e lowe bo d o e e

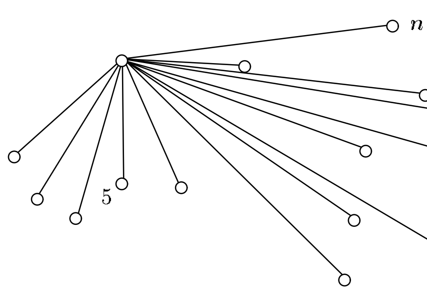


i . . n x pl lls l b n

.

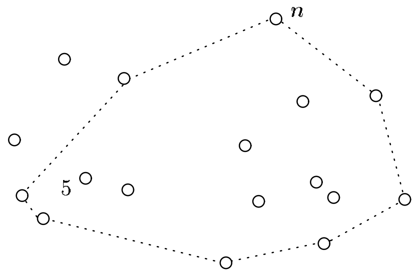
r e co c e o m po c we ve m po eq ll p ce o c cle o d d e o e m po o c cle o d co m + w e e ll po ve v l e I ece we pe b e po l l o o po e o e e l e ee F e I odd pl ce dd o l po clo e o e ce e e e e e e o e p le de p o po o e o e c cle o cell e o e wo po o e o e c cle c ev e de ed bo d ⊔

Fo e ppe bo d we p e e e ve co c o e de o e e cloc w e c c l eq e ce o po w e ve e ve ce o e co ve ll o e p o e co c o e e po o d wo co ec ve po o w e e e l e e e be wee e wo po c lled *r i g dg* e d ce pe o o e po le po d be ed p o co ec ve po o e e o e eq e ce ob ed b weep c o ed d p o w c pl c l e o e e bo o o ee F e e e I e o ee d U e c eve 5 po e p co ve o 5 we ded ce 5 d Fo v l e o 7 we e e ve pp o c



i .3. T p n s n n n n n n l s p y n

e e c e o e e e wo b c o co e I e ce we loo o po o cell e e po d e e bo d e ed e e ed e o e e e o Ob e ve lw I e p e e o o e w e e o e e bo o o co ed I co ve e l o e p d we e do e ee F e

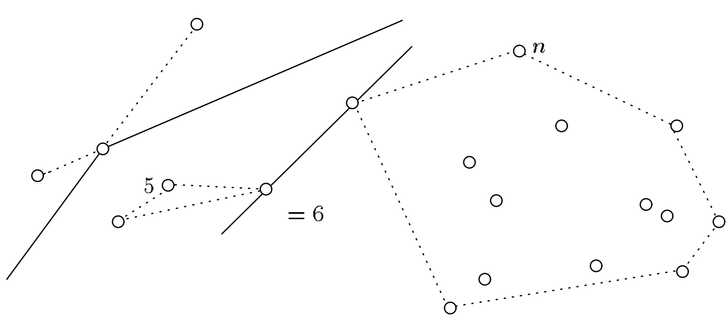


i . . T s s n x

ppo e o co ve e eco d po b l ope e o 7 po
b e e we w ll d e e o 7 po b e c c be
deco po ed o o cell e pec vel o o e o e cell d
o e cell e e ed e o d e co ec ve o
we c e e ed e o e e e o e
w ll de l e deco po o p oce o 7 po b e Ob e ve e
be o cell p od ce b e ve p oce 5

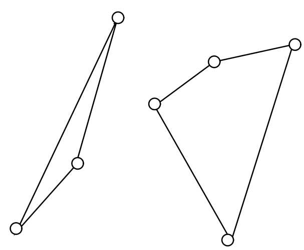
Co de e l e p o e po d p o e
pl e o e o pl e l e e elve po co eq e ce o e
c e e o o po e e p co ve o w
o e o e e e o co ll o ee F e 5

e 5 d 5 Co de e e co o
e l e o d c o e l pl e bo ded b
co d e p be po d d be po d
e e wo po p be c lled d ee l o F e 5 e e e
wo p c e o co de



i . . fin p p n s n n \ ∪)

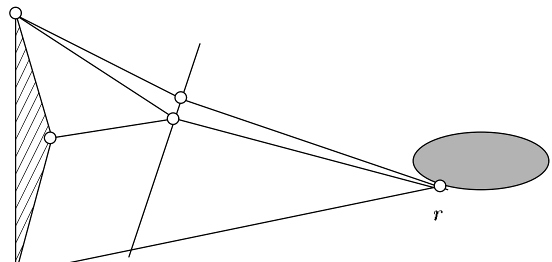
C e e l e o d doe o e ec e l e e e be wee
p d e *p* e p co ve o Fo co ve e ce we e e e
po $_2$ $_3$ e ppe cloc w e o de o e bo d o
e cell w w e e $_3$ *p* ee F e I l e o doe
o e ec e ed e e we ob cell o dd o *p* d
e ed e o e e e o $_3$ O e w e le be e
l e e e c e d ec ed c o ed $_i$ d p o
$_i$ e ec e ed e I / we e + I e l e o $_i$ $_j$ l o
e ec w e ed e e we e ee cell $_i$ $_j$ $_2$ $_3$ d
d $_3$ e ed e o e e e o I $_i$ $_j$ doe o e ec
w e ed e we e $_i$ $_j$ $_2$ $_3$ d w e ed e $_3$
I e l e o $_3$ ece l e ec e ed e I c e we
ob wo cell 5 cell $_2$ $_3$ d d e e ed e
o e e e o



i . .

C e B e l e o d e ec e l e e e be wee *p* d
e be e po clo e o e e e *p* De o e e cloc w e le o
o e po b _ I _ co ve e we ob
ee cell *p* d w e ed e *p* o e e e o
O e w e we e e e pe o *p* d co ve we e w
d *p* e ed e o e e e o *p* oweve *p*
o be co ve o eve be ple we eed o e e ew
o e b c e
e e e e e po ble c e cco d o w c o *p* eq l o

p pl e *p* co ve o o o e eed o be do e
e *r* c e le _ *r* zed ce e l e
o d e ec e l e e e be wee *p* d we ded ce
p ee F e 7 e ob cell *r* d d we e *r* e
ed e o e e e o I _ *p* co ve e we l o e
e 5 cell *p* O e o e d _ *p* co ve we e e 5 cell
r *p*



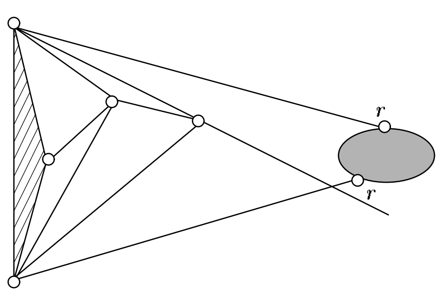
i . . _) s n x _) n x

e r r c _ r d _ r e d e pec vel e co de e o cco d o e po o o p o e p ee F e

p e ob e wo cell d p d e e ed e o e e e o

b p wo o e ee de ed cell e d p le o e o pr o r be co ve I e o pr co ve c oo e e d cell d r e ed e o e e e o

O e w e c oo e r e d cell d r o e ed e c p e ob ee cell p d r p d e ed e r e ed e o e e e o



i . . n p n

co c o p ove e ollow le

. 5

Co b le d p ove o eo e

Di io

e ve ppe d lowe bo d o v o o co ve p o e e ll b l p e d be wee e ppe d lowe bo d F o o po o v ew o e co pl c ed l o be ble o p ove e ppe bo d o ew b we co jec e e co ec bo d e lowe bo d l o e o p ove ee o be o e d c l o p ove po e e ee o be co ec o w e l e e o w e e e o o Ob e ve o ll b el pl e e e e o

A owe e e t

e wo ld l e o e e e ee o e c e l e d o e c p d e elp l co e

Re e e

s n pp p C p n l s p y n x s bs s p n s n *P* *s* *M S* *s* *G* 6 – 6 5
s b n l p bl s n y *s* *M* *s* : 6–5 0
s n s b n l p bl n y *s* *M* : 6 – 0 5
b n x F¨n n b n n n n n *M* : 6–
5 n s n p y n x ns *d M* 6: –
6 s l n n n n s p y n x 6 ns *S* : 60– 6
b n p n n n x p ly ns *s* *M* 6 : – 6
b n n p n s s n sj n n x p ly p s *G* : –

V m s P h d

J n n n e n nger n

ar r Sc c
rs rs
w r sw ck S

bs ac W r s a s l a rac cal a a s r c r r s r a c ssar l c) l r w c ca a q r s r ac c l r r l a ar a r l r a q r s ar a sw r a l ar s z c l) s ac a r r c ss r s a a s r c r ar l ar s z W als r s a r ar a a s r c r r la ar ra s w c s s r a l ca a s

o o oy o

is p per s ies er n r gener i i n e ing pr e in R : i en si p e p g n ns r s r re i gi en er r c ine seg en ing in e pie es se n ries re n e ges re rns e re e p g n e c A ge e ri e s in is p per re ss e e n nneg i e r in es

n s s i n is pr p se r n e p g ns: nsi er e p g n i e ges e e rien e in ise r er s r ing e es er e s es r in e e e e e signe re e r pe i efine e e ge e n i s pr e i n n e x is e re e is neg i e iff e e ge e is e ge ere e in eri r e p g n ies e ig er x r in e e e ge e e e pr e i e re e e re is si p $\sum$ e pp se e r c s i s en p in s n e e ges e n e ere $\leq$ e e n e e e p r i ns e n e e c e re e c is $\sum$ e e e c

s ring s $\sum$ e r e n p e $\sum$ s s in ns n i e n s e ge :

r *G p ly w h r s h r s d s ru ur h r pr pr ss r ur h r u ry h rd c s*

r n n n e p g ns es ri es s r re re ires g prepr essing i e n gi e e re e p g n e er r in g i e e en in e rs n i e i e er r esn' r ss n e ge e p g n e g ri se r e re r s i ifi i n e s s er pp i i ns e s r re

ac n c e ed s arc s r S ra 97 89
a e c e ed

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 156-159, 2001.

o y

e n is ss ree i ensi n gener i i n e p g n pr e is in eres ing n e e i e s presen e ere e se e en e s r re ig er i ensi ns g e n n n pp i i ns s s r re

n r er gener i e e pr e i ensi ns e firs nee gener i e e n ep r n r se ing e e p e r n i er i es A *u ry sur* S is p e r s r e sep r ing in e pie es n e n r S ies n e s r e e is i p ies e es rip i n S i e e s s rge s e n er e ges r sse e n r S n e er si e is is s i e e s er n A e er n s r e s i re rn e e e p r i n ing e s e i s e ing:

r *G p lyh dr s h r s s ru ur h w h pr pr ss w ll sw r lu u r s l r h s h u ry sur*

e i e i e si i r e i ensi n se irs e efine e signe *pr lu v* p g n e e e e p e r n efine n i s pr e i n n e p ne r e v i e neg e i e insi e is e e e n en e e presse s e s r e s v

in r er ns er e er r s r e s e nee

i p e e s v r e s s ri e s

ii e pr e i e e e pper p r i ns e e s re s n

iii s r e pr e i e e e p g ns r ing s

e e e si e e es rip i n s e n e si p e e e r iii in i e

s ring ri n e i ensi n s r re e re in e e i is p ssi e fin e pr e i e e e in i e is n e ie e rep ing e pr e i e re n e ge i e e pris efine e e ge e e ge's pr e i n n e p ne n n fi e p in n e in erse i n e p ne n e p ne n ining e e in e e n er e s is ess n e n p e ii in i e

e n s re ining is p e i n r er e nee s e prepr essing: p e e pr e i e e e e e e e i is e r is p in n e i re e p e r s p n r gr p i e e e e i i s pr e i e e e n n re e r pr e pr e r e e e p n r gr p s e efine er p n r gr p e se ire e e ges en re e r p n r gr p sep r e nne e p nen e er p nen r e res e gr p e e ges er re e presen e in ise r er e er p nen

r *G dd pl r r ph wh r w h s r s s d ry h r s s ru ur h w h pr pr ss w ll r ur h su h w h s h s u ry p u ry u l r h u r d s h u*

e n r er s r e r e p e r n e r efines er r e p n r gr p n e ns er re rne e p n r gr p s r re i ns er e p in i e pre i s se i n

e is s r re is in epen en in eres n e se in re s s s r gr p n p er ne r s ne p ssi e ppi i n e in ge e ri r ing ere e ser ir es n re n ne r r s ess ge i e se firs p e e si e r er s is i e se e e r nge see i i ies i in ep e p r e ers ee e i firs s e s ig si p er pr e : efine er ir i e ir r er e is in ern ing se en e er i es n e ges in n e e ing p n r gr p

r *G dd pl r r ph wh r w h s r s s d ry h r s s ru ur h w h pr pr ss w ll r ur h su h w h s h s s d u ry r u l r h s h r u*

r r c in irs ns r ire e p n e p ne isi s e er e e s n e n s r s n en s in e si e e is n e ne i ns r ing e eri n p sp nning ree r e gr p r e e ssign e eig e s e p in n in e r n p in n efine e e s e eig p in s r e eginning e p

C nsi er e er e s ise rien e e i e e er e ge e es ire e e ges r i e ges eig s n e p r e e ge r sses i s in erse i n i s eig i i is rig rn r n i i is e rn

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Sum of Edge Lengths of a Graph Drawn on a Convex Polygon

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Abstract. Let $x_0, x_1, ..., x_{n-1}$ be vertices of a convex n-gon P in the plane, where, x_0x_1, x_1x_2, $\ldots$, $x_{n-2}x_{n-1}$, and $x_{n-1}x_0$ are edges of P. Let $G = (N,\ E)$ be a graph, such that $N = \{0,\ 1,\ \ldots,\ n-1\}$. Consider a graph drawing of G such that each vertex $i \in N$ is represented by x_i and each edge $(i, j) \in E$ is drawn by a straight line segment. Denote the sum of lengths of graph edges in such drawing by $S_P(G)$. If $S_P(G) \leq S_P(G')$ for any convex n-gon P, then we write as $G \preceq_l G'$. This paper shows two necessary and sufficient conditions of $G \preceq_l G'$. Moreover, these conditions can be calculated in polynomial time for any given G and G'.

1 Introduction

Let $x_0, x_1, ..., x_{n-1}$ be vertices of a convex n-gon in the plane (each internal angle may be equal to π), where, x_0x_1, x_1x_2, $\ldots$, $x_{n-2}x_{n-1}$, and $x_{n-1}x_0$ are edges of the n-gon. Denote the length of the line segment x_ix_j by $d(i, j)$. i mod n denotes i' such that $i \equiv i' \pmod{n}$ and $0 \leq i' \leq n-1$.

Let $G = (N,\ E)$ be a graph with a vertex set $N = \{0,\ 1,\ \ldots,\ n-1\}$ and an edge set E. Parallel edges and self-loops are permitted in G. In $G = (N,\ E)$, E may be denoted by $E(G)$. In this paper, a vertex set of each graph is fixed to $N = \{0,\ 1,\ \ldots,\ n-1\}$. Define a length of G with respect to an n-gon P as

$$S_P(G) := \sum_{(i,j) \in E(G)} d(i, j).$$

$S_P(G)$ can be regarded as a sum of edge length of a graph G drawn in the plane such that each vertex of G is equal to a corresponding vertex of P and each edge of G is written by a straight line segment. Graph drawing has recently become a very important research area and the sum of edge lengths is one of the crucial criteria for evaluating drawing methods[1].

A Partial-Order "$\preceq_l$" Based on $S_P(G)$.
We introduce a partial-order "$\preceq_l$" as follows. Let G and G' be two graphs. If $S_P(G) \leq S_P(G')$ for any convex polygon P, then $G \preceq_l G'$ ("l" means length).

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If $G \preceq_l G'$ and $G \neq G'$, then $G \prec_l G'$. $\preceq_l$ is clearly a partial-order.

For any two subsets $X, Y \subseteq V$, $E(X,\ Y;\ G)$ denotes an edge set between X and Y, i.e.,

$$E(X,\ Y;\ G) := \{(i,j) \in E \mid i \in X,\ j \in Y\}.$$

For $i, j \in N$, define

$$N[i,j] := \begin{cases} \{i,\ i+1,\ \ldots,\ j\}, & \text{if } i \leq j, \\ \{i,\ i+1,\ \ldots,\ n-1,\ 0,\ 1,\ \ldots,\ j\}, & \text{if } i > j. \end{cases}$$

If $N[i,j]$ is a proper subset of N, $N[i,j]$ is called a *neighbor-cut*.

If there is no neighbor-cut $N[i,j]$ such that $E(N[i,j];\ G) = \emptyset$, then G is called *neighbor-connected*. We define

$$E_q := \{(i,\ i+q \bmod n) \mid i \in N\}$$

for each integer $0 \leq q \leq \lfloor n/2 \rfloor$. $G_q := (N,\ E_q)$. G_q is a 2-regular graph.

The authors have already presented the following properties[2,3].

Theorem A
(1) $S_P(G_q) \prec_l S_P(G_{q+1})$ for $q = 0,\ 1,\ \ldots,\ \lfloor n/2 \rfloor - 1$.
(2) For any 2-regular graph $G(\neq G_{\lfloor n/2 \rfloor})$, $G \prec_l G_{\lfloor n/2 \rfloor}$.
(3) If $G(\neq G_1)$ is a neighbor-connected 2-regular graph, $G_1 \prec_l G$. □

In this paper, we present a general rule on $S_P(G)$, which includes Theorem A. For explaining this rule, we give some notations.

$E(X,\ V-X;\ G)$ can be also represented as $E(X;\ G)$ for notational simplicity. $|E(X,\ Y;\ G)|$ and $|E(X;\ G)|$ may be written as $c(X,\ Y;\ G)$ and $c(X;\ G)$, respectively. A singleton set $\{x\}$ may be simply written as x. $N(i,j) := N[i,j] - \{i,j\}$, $N(i,j] := N[i,j] - \{i\}$, $N[i,j) := N[i,j] - \{j\}$.

Cross-Operation and a Partial-Order "$\preceq_o$".
If two distinct edges $(i,j), (h,k) \in E$ satisfy that $h, k \in N(i,j)$ or $h, k \in N(j,i)$, then we say that (i,j) and (h,k) are *separated*. If "$h \in N(i,j)$ and $k \in N(j,i)$" or "$k \in N(i,j)$ and $h \in N(j,i)$," then we say that (i,j) and (h,k) are *crossing*.

Let (i,j) and (h,k) be a separated pair of edges. Without loss of generality, we assume that $i \in N(h,j]$ and $k \in N(j,h]$ (see Figure 1 (a)). By deleting (i,j) and (h,k) from $E(G)$ and putting (i,k) and (h,j) in $E(G)$, a new graph $G' = (N, E')$ is obtained (see Figure 1). This operation is called a *cross-operation*.

If G' can be obtained from G by applying a sequence of cross-operations, then $G \preceq_o G'$ ("o" means operation). $G \prec_o G'$ means $G \preceq_o G'$ and $G \neq G'$.

A Partial-Order '$\preceq_c$" Based on the Size of Neighbor-Cuts.
If $c(N[i,j];\ G) \leq c(N[i,j];\ G')$ for every neighbor-cut $N[i,j]$, then $G \preceq_c G'$ ("c" means cut). We will show in Corollary 1 that if $G \preceq_c G'$ and $G' \preceq_c G$, then

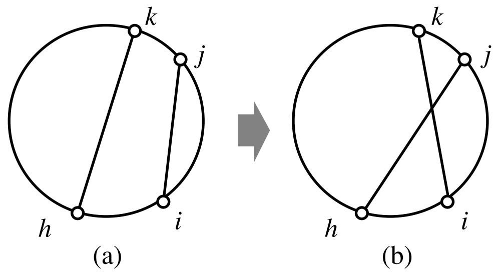

Fig. 1. Cross-operation.

$G = G'$. Thus we can say $G \prec_c G'$ if $G \preceq_c G'$ and $G \neq G'$.

Main Theorem *Three partial orders $\preceq_l$, $\preceq_o$, and $\preceq_c$ are equivalent for any pair of graphs $G = (N, E)$ and $G' = (N, E')$ with $|E| = |E'|$. That is, each one of $G \preceq_l G'$, $G \preceq_o G'$, and $G \preceq_c G'$ implies the others.*

Theorem A is a corollary of the main theorem.

2 Proof

Lemma 1. *If $G \preceq_o G'$, then $G \preceq_l G'$.*

Proof: It is clear from the triangle inequality. □

Lemma 2. *If $G \preceq_l G'$, then $G \preceq_c G'$.*

Proof: Suppose that $G \preceq_c G'$ does not hold, i.e., there are $i, j \in N$ such that $c(N[i, j];\ G) > c(N[i, j];\ G')$. We construct a polygon P satisfying $S_P(G) > S_P(G')$ as follows. $X = \{x_k \mid k \in N[i, j]\}$ and $Y = \{x_k \mid k \in N(j, i)\}$. Put all vertices $x \in X$ in a circle whose center is $(0, 0)$ and radius is r. Let $p > 0$ be a real number. Put all vertices $x \in Y$ in a circle whose center is $(p, 0)$ and radius is r. We can locate all vertices satisfying the above conditions and convexity for each r and p. By letting p be far larger than r, $S_P(G) > S_P(G')$. □

Lemma 3. *For any pair of graphs $G = (N, E)$ and $G' = (N, E')$ with $|E(G)| = |E(G')|$, $G \preceq_c G'$ implies $G \preceq_o G'$.*

For proving this lemma, we need some lemmas as follows.

Lemma 4. *Let $G = (N,\ E)$ and $G' = (N,\ E')$ be two graphs. For any neighbor-cut $N[i, j]$, $c(N[i, j];\ G) - c(N[i, j];\ G')$ is even if and only if $\sum\{c(k;\ G) - c(k;\ G') \mid k \in N[i, j]\}$ is even.*

Proof:

$$\sum_{k\in N[i,j]} c(k;\ G) = c(N[i,j];\ G) + 2c(N[i,j],\ N[i,j];\ G) \tag{1}$$

$$\sum_{k\in N[i,j]} c(k;\ G') = c(N[i,j];\ G') + 2c(N[i,j],\ N[i,j];\ G') \tag{2}$$

From (1)−(2), the statement is assured. □

For $i, j \in N$, a graph obtained from $G = (N, E)$ by contracting $N(j, i)$ to a vertex is denoted by $G[i, j]$. For an integer k $(0 \le k \le n)$,

$$N_{=k} := \{(i,j) \mid i,j \in N,\ |N[i,j]| = k\},$$
$$N_{<k} := \{(i,j) \mid i,j \in N,\ |N[i,j]| < k\},$$
$$N_{\le k} := \{(i,j) \mid i,j \in N,\ |N[i,j]| \le k\}.$$

Lemma 5. *Let $0 \le k \le n-1$ be an integer. If $c(N[i,j]; G) = c(N[i,j]; G')$ for all $(i,j) \in N_{\le k}$, then $G[i,j] = G'[i,j]$ for all $(i,j) \in N_{\le k}$.*

Proof: We use induction. If $k = 0$ or 1, it is clear. Assume that for an $h \ge 2$ if $k < h$, the statement is correct. Further assume that $c(N[i,j]; G) = c(N[i,j]; G')$ for all $(i,j) \in N_{\le h}$. From these assumptions, we derive that $G[i,j] = G'[i,j]$ for all $(i,j) \in N_{\le h}$.

Consider $(i,j) \in N_{=h}$. If $c(i,j; G) = c(i,j; G')$, then $G[i,j] = G'[i,j]$. Then we assume that $c(i,j; G) > c(i,j; G')$ without loss of generality. From $c(i; G) = c(i; G')$ and $c(i, N(i,j); G) = c(i, N(i,j); G')$,

$$c(i, N(j,i); G) < c(i, N(j,i); G'). \tag{3}$$

Similarly,

$$c(j, N(j,i); G) < c(j, N(j,i); G'). \tag{4}$$

From the assumption,

$$c(N(i,j), N[j,i]; G) = c(N(i,j), N[j,i]; G'),$$
$$c(N(i,j), i; G) = c(N(i,j), i; G'),\ \text{and}$$
$$c(N(i,j), j; G) = c(N(i,j), j; G'),$$

hence

$$c(N(i,j), N(j,i); G) = c(N(i,j), N(j,i); G'). \tag{5}$$

From (3), (4), and (5), $c(N[i,j]; G) < c(N[i,j]; G')$, contradicting the assumption. □

Corollary 1. *If $G \preceq_c G'$ and $G' \preceq_c G$, then $G = G'$.*

Proof: By letting $k = n - 1$ in Lemma 5, the statement can be obtained. □

Lemma 6. *Let k $(2 \leq k \leq n-1)$ be an integer. Let i, j be integers such as $(i,j) \in N_{=k}$. If $c(N[i'\ j']; G) = c(N[i', j']; G')$ for all $(i', j') \in N_{<k}$ and $c(N[i,j]; G) < c(N[i,j]; G')$, then $c(i,j;G) > c(i,j;G')$.*

Proof: From Lemma 5, we obtain

$$c(i, N(i,j); G) = c(i, N(i,j); G') \text{ and}$$
$$c(j, N(i,j); G) = c(j, N(i,j); G').$$

From the assumption,

$$c(N(i,j); G) = c(N(i,j); G').$$

Thus,

$$c(N(i,j), N(j,i); G) = c(N(i,j), N(j,i); G').$$

By considering $c(N[i,j]; G) < c(N[i,j]; G')$,

$$c(i, N(j,i); G) < c(i, N(j,i); G') \text{ or } c(j, N(j,i); G) < c(j, N(j,i); G')$$

hold. Without loss of generality,

$$c(i, N(j,i); G) < c(i, N(j,i); G').$$

We have

$$c(i; G) = c(i; G') \text{ and } c(i, N(i,j); G) = c(i, N(i,j); G'),$$

thus $c(i,j;G) > c(i,j;G')$ is obtained. □

Now, we can prove Lemma 3.

Proof of Lemma 3: Assume that $G \preceq_c G'$. Let k $(0 \leq k \leq n-1)$ be a largest integer satisfying that

$$c(N[i,j]; G) = c(N[i,j]; G') \text{ for all } (i,j) \in N_{\leq k}.$$

From Lemma 5,

$$G[i,j] = G'[i,j] \text{ for all } (i,j) \in N_{\leq k}.$$

Then if $k = n - 1$, the statement is trivial.

Otherwise, let $i_0, j_0 \in \{N \mid |N[i_0, j_0]| = k+1\}$ be a pair such that $c(N[i_0, j_0]; G) \neq c(N[i_0, j_0]; G')$ (see Figure 2).

From $G \preceq_c G'$,

$$c(N[i_0, j_0]; G) < c(N[i_0, j_0]; G').$$

From Lemma 6, $c(i_0, j_0; G) > c(i_0, j_0; G')$, hence $(i_0, j_0) \in E(G)$.

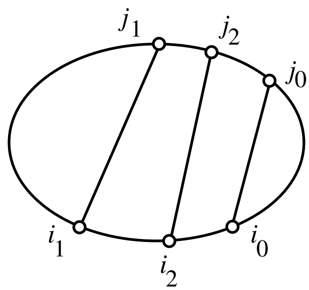

Fig. 2. Relation of i_0, j_0, i_1, j_1, i_2, and j_2.

$c(N[i_0, j_0]; G) < c(N[i_0, j_0]; G')$ is equivalent to $c(N(j_0, i_0); G) < c(N(j_0, i_0); G')$. Let i_1 be an integer such that

$$c(N(j_0, i]; G) < c(N(j_0, i]; G') \text{ for } i \in N[i_1, i_0) \text{ and}$$
$$c(N(j_0, i_1); G) = c(N(j_0, i_1); G').$$

(From the assumption, there exists such an i_1.) It follows that there exists an edge $(i_1, j_1) \in E(G)$ such that $j_1 \in N(j_0, i_1)$. Let $i_2 \in N[i_1, i_0)$ and $j_2 \in N(j_0, j_1]$ be integers such that $(i_2, j_2) \in E(G)$,

$$c(N[i_1, i_0),\ N(j_0, j_2); G) = 0 \tag{6}$$

and $c(N(i_2, i_0), j_2; G) = 0$. Note that i_2 and j_2 may be equal to i_1 and j_1, respectively.

We will show that $c(N[j, i]; G) < c(N[j, i]; G')$ for any $i \in N[i_2, i_0)$ and $j \in N(j_0, j_2]$ as follows. Assume that there are $i' \in N[i_2, i_0)$ and $j' \in N(j_0, j_2]$ such that $c(N[j', i']; G) = c(N[j', i']; G')$. By considering

$$c(N(j_0, i_1); G) = c(N(j_0, i_1); G'),$$
$$c(N(j_0, i']; G) < c(N(j_0, i']; G'), \text{ and}$$
$$c(N[j', i_1); G) \le c(N[j', i_1); G'),$$

we obtain that $c(N[i_1, i'], N(j_0, j'); G) > 0$, contradicting (6). Therefore,

$$c(N[j, i]; G) < c(N[j, i]; G') \text{ for any } i \in N[i_2, i_0) \text{ and } j \in N(j_0, j_2]. \tag{7}$$

From the assumption, $c(i; G) = c(i; G')$ for all $i \in N$. Thus from Lemma 4, expression (7) can be rewritten as

$$c(N[j, i]; G) \le c(N[j, i]; G') - 2 \text{ for any } i \in N[i_2, i_0) \text{ and } j \in N(j_0, j_2]. \tag{8}$$

Denote a graph obtained by applying cross-operation $(i_0, j_0; j_2, i_2)$ to G by G''. Clearly, $G \prec_c G''$. From (8), $G'' \preceq_c G'$. Thus by applying the above operation recursively, we can get a sequence of cross-operations for modifying G to G', i.e., $G \preceq_o G'$. □

Proof of Main Theorem: It is clear from Lemmas 1, 2, and 3. □

3 Concluding Remarks

This paper shows that three partial-orders $\preceq_l$, $\preceq_o$, and $\preceq_c$ are equivalent. For investigating $G \preceq_c G'$, only neighbor-cuts are tested, thus it can be determined in polynomial time. Therefore, we can solve a problem of determining whether or not $S_P(G) \leq S_P(G')$ for any convex polygon P for given two labeled graphs G and G' with $|E(G)| = |E(G')|$ in polynomial time. Moreover, if $G \preceq_c G'$, we can find a sequence of cross-operations for modifying G to G' by using the discussion of the proof of Lemma 3 in polynomial time.

In this paper, Euclidean distance is used. However, for any distance (for example, L_k distance) in which the triangle inequality holds, the same results can be obtained.

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in e r v n v v v v n v v s is p e e ip r i e gr p r v s v v n n v e ng i e n ining n s v n is p e e gr p in e is D gr p e re

O O

e *r* ◦ gr p s n is efine s e gr p ine ing ne p n pies n en ining e er e e er er e in e p

r iti n r n) *r d r phs d w h* ◦ *s B r ph d ly s p r r ph d s UB r ph*

r e e ip r i e gr p i p r i e se s n N en is D gr p i n pper erne n er erne N se n e v is D gr p r gr p ◦ is D gr p C n erse e ◦ is D gr p i n pper erne ◦ n er erne ◦ en e e e ing i

i L C , *l u* ◦ *su h h* ◦ ◦ *d* C , *r* v v C , *h*

r Cl 1) in e v N v N en N v N

e ss e ◦ ◦ n v ◦ ◦ in e is nne e n v is en n er er e ere e is s i e C s v C ◦ C n ◦ C e e p rresp n ing v en r e er e x v is en x s v n v in e v is er e ◦ v v is n p e e ip r i e gr p i n r i s e n i i ns e re ere re ◦ ◦ n is ip r i e gr p in e v is er e ◦ n v ◦ ◦ v v is D gr p n is gr p

4 O

e C *r s pr du* gr p s n is e gr p i er e se ere e se n is e se C r esi n pr n e ges efine s s: v is en v i ei er n v is en v in r is en in n v v en is e ni n pies e n e ing s e efini i n

ct) r gr p s n i n is ip r i e gr p i n n i n re ip r i e gr p s

ct) e C e i e v C en r x C x r r x C v

r iti n rt i n r d ct) *r d r phs d w h d s B r ph d ly h d r p r r phs*

r n re ip r i e gr p s en is ip r i e gr p n D gr p
C n erse e e D gr p i n pper erne n er erne e ss e n *v* in e n ere e is e ges *v* *v* n *v* ere e is *v* *v* n *v* n *v* re en *v* n *v* n n re en *v* n s *v* is en i n r i s e efini i n ere re n is ip r i e gr p n re ip r i e gr p s

5 o o o

e *p s* gr p s n is e gr p i e er e se n e ges efine s s: *v* is en *v* i ei er is en in r n *v* is en *v* in

r iti n iti n) *I s B r ph d s pl r ph h s B r ph*

r e C C C e D e ge i e er in e is p e e gr p C is p e e s gr p e fi β r n pper erne $_s$ n er erne ; n N β ; C C C is n e ge ers r er re r e C ere e is n β N s β C n β C r in e e s e n er i es n ining r e v N v v e s e n er i es n ining v s is D e ge i e er

6 o o

e *r l pr du* gr p s n is e gr p i e er e se n e ges efine s s: *v* is en *v* i ne e ing s: is en in n *v* is en *v* in n *v* is en *v* in is en in n *v* *v* r p se s n e *pr du* n is e p se n e C r esi n pr e e e en se n e e e en se s x $\leq$ v i $\leq$ v n $x \leq$ ese efini i ns e in e ing r D gr p s n is D gr p p se ere n re p se s se D gr p s re n e e e ing res

r iti n r r d ct) *r B r phs d s B r ph*

O O

e u gr p s n is e gr p i e er e se n e ges efine s s: v is en v i is en in n v is en v in

r d r phs d s p r r ph d ly r s p r r ph

r e ss e n re n ip r i e gr p s e $C : x$ v v x e n e n : e n e en in e se x v v v v x is n e n e se x v v v v v v v v v v x is n e s is n ip r i e gr p e e ip r i e gr p i p r i e se s n N en N n is ip r i e gr p i p r i e se s n N

L B r ph w h upp r r l d l w r r l I c h c

r e ss e c n c en ere e is x n s x c x c s x n is en x n in e is D gr p N N x is p e e s gr p is en i n r i s e efini i n n n i ns

L d d r phs s B r ph d ly s p r r ph

r is ip r i e gr p en is D gr p C n erse e e D gr p i n pper erne n er erne e ss e e x s x

is en n s is en
en x is en n x e
in e x is n en x n e e n r i i n

e n e in e ing res

r iti n 6 n ncti n) *r d r phs d s*
B r ph d ly r s p r r ph

e *ddl r ph* gr p is e in erse i n gr p $\mathcal{F}$
v v v ere v v v is e er e se
n is e e ge se is n
er i es n β in is en i n n i β n is
en β r β n is in i en β

L d r ph d s B r ph w h upp r
r l d l w r r l L v r I v
h e v

r e ss e v n *e* v in e v ;
N v $_M$) v r s i e n v ; N v e
n ss e *vx* e ngs n v is
en v in n is n en *vx* n v in en ere
e is β n in e is en v n is n
en v e e n r i i n

L d r ph d s B r ph w h up
p r r l d l w r r l I v h v

r e ss e v n v *e*
en is n en v in s v
e i i r *e* v e v
en ere e is $v\beta$ in e v en $v\beta$
is en $v\beta$ n β en is en v in i is
n r i i n

r iti n Midd r) *r d r ph s B*
r ph d ly G s y l r p h

r e e n e en e r p en is D gr p
C n erse e ss e is D gr p s er e v i
e v en v ; is en v $_M$) is p e e s gr p
i e s er i es i n r i s e $_M$)
is ip r i e gr p e s e i egree is s
n is e r p is n e en $_M$) s n
e s is n e en e r p

9 o

e *l r ph* gr p is e gr p i e er e se n er i es n β in is en i n n i β n is en β r β n is en β r β n is in i en β

6 *L d r ph d s B r ph w h upp r r l d l w r r l L v r I e v h v r ll N v*

r e ss e *e v* n *v* in e is en *v v* r n *v* r *v* n n e se *v* ere e is n er i es en *v* n *v* n e se n *v v* n *v* re en *v* n *v* is n en ere is n er en er e *v* s *v* n is en en *v* n re en e er is n en *v* n e se *v* n sin e *v* n *v* re en *v* in ere is n er en er e *v v* is en n e er se e e n r i i n *v* r *N v* i *e v*

r iti n 8 t r) *r d r ph w h r s s B r ph*

r ere e is s er e *v* i *e v* e *x* e en *v* in en *vx* is en *v* in n *vx v* e 6 s ere e is er i es en *vx* n *v* in e 6 *v* r er e en *v* ere re e e n ses: *v* n *x v* n *x* e er *v* is n en *x* ere e is s n i e n ining e e ge *vx* en e is n D gr p

1 za a ar ra al ra s a ra rsa l r c E r a S c **1** 19)117 1
l ra a ar all r r s r al a r cs a & S s Sc c s **10** 198)
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4 Era O awa a s c a O r ra s w r s c r a s ra s r cal r Sc c)) 19
Era a s c a O r ra s w s c l s ar als r ra s scr a a cs **1** 1998) 1 1 9
a a a a s ra ra rsa l a c c l ra a ra scr a a cs **1** 197) 47

7 arar ra r s W sl 19 9)
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ra s r r)
9 c rr s a asla sk ra s a ar all r r s r al
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1 O awa a s c a O r ra s w r s c l ra s
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z d c d P s T s P s h P

A s s i K ne n K n

[1] ar r Sc c a ca E r
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bs ac W c s r ll w r l n 1 a
rs a B w s s s n r s a
n l s la r s c l s c a r s B
l sa l $n = n_1$ n_2 $\cdots$ n a r ar
n s c a 1 n r r 1 w wa ar
B s s s s $_1$ $_2$ $\cdots$ a sa s ll w
w c s) c) ∩ c) = r all 1 w r
c) s c ll a) ac c a s ac l
n r s a n l s r r 1
W s all r a a ar s s cas w r)
n 8 a 1 n $n/$ r r 1 a) $n_1 = n_2 = \cdots =$
n $_1 =$ a n = 1

O O

r se p in s in e p ne e en e n e n e
i is e s es n e se n ining e s nsi er e ing
pr e :

r l 1 e R n e n is in se s re p in s n e
p in s in e p ne respe i e s n ree p in s R ie n e
s e ine in p si i e in eger p r i i ns r i R
n e p r i i ne in q is in s se s s is e
ing n i i ns:

n n r ≤ ≤ q; n
e n ins e re p in s n e p in s

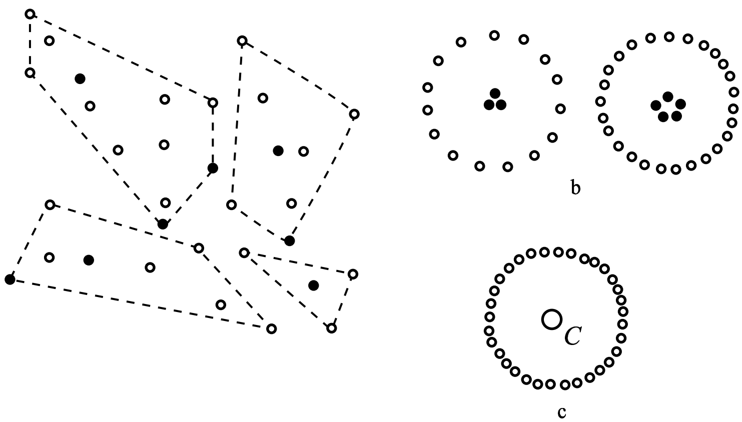
R n e p r i i ne in q s se s in e e en e s
R is p r i i ne in *l d su s s* r R s
l d p r

ig re gi es n e p r i i n ig re s s nfig
r i ns R ing n n e p r i i n r n n e p r i i n
respe i e n e is en e s n e p r i i ns n e s n e
i s n e p r i i ns e is en ere e is ines p r i i n R

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 176-186, 2001.

in s se s s is ing e e n i i ns n e er ere e is n s ines

si i r rg en gi en e i en e n e si gi e nfig r i ns R ing n n e p r i i ns s s: e C n C e ir es i e s e en er in e p ne s e r i s C is s er n C n e s ien rge in eger en e ni r p e re p in s n e p in s n e n ries C n C respe i e see ig re



g 1 a) 1) ala c ar) ra s a 1) ala c ar a) ala c ar r s c l c) c ra a n_1 n_2) ala c ar w $n/$ n_1 n

r g is p per e R n en e is in se s re p in s n e p in s in e p ne respe i e s n ree p in s R ie n e s e ine e s gi e s e p r s e ing e re s sin e eir p e e pr s re i e ng n r er e i s

r *L* $\leq$ $\leq$ $\leq$ *d* $\leq q$ *rs L R d d s s s r d p s d lu p s h pl r sp ly h r ry r p r su h h* $\leq$ $\leq$ *r ry* $\leq$ $\leq q$ *R h s l d p r*

r *L* $\leq$ *dd r d r L R d d s s s r d p s d lu p s h pl r sp ly h R h s l d p r*

e e res s n n e p e gi en er e pr p se e ing n e re

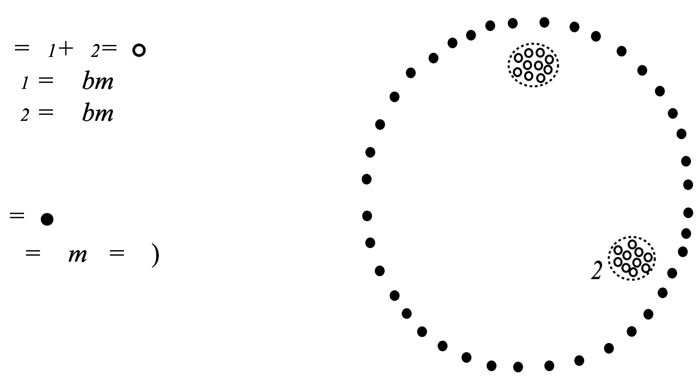
C *ur* *1* e n *q* e in egers e *R* n e is in se s re p in s n e p in s in e p ne respe i e e e n in eger p r i i n s ≤ ≤ r e er ≤ ≤ *q* en *R* n e p r i i ne in *q* is in s se s s i n n r ≤ ≤ *q*; n ii e n ins e re p in s n e p in s

e n e re is r e i ei er n ≤ r n e re s n re er e n e re is r e in e se ere is e ing e re s is e re s p r i pr e n 6 n p e e pr e esp ni Kir p ri n n e in e r n Y n i in epen en

r *L* *R* *d* *w* *d s* *s* *s* *r d p* *s* *d* *lu* *p* *s* *h* *pl* *r sp* *ly* *wh* *r* *d* *h* *R* *p* *r* *d* *d s* *su* *s* *s* *s* *h* *)* n n *r* *ll* ≤ ≤ *d* *)* *ry* *s* *ly* *r d p* *s* *d* *lu* *p* *s*

er re e res s n e n in n i e i n e p r i i n n e se s in e p ne

e n e is se i n i n e p e i s s e n i i n ≤ ≤ r e er ≤ ≤ *q* in C n e re r er si i r n i i n is ne ess r n C n e re es n n er e n i i n ≤ ≤ r e er ≤ ≤ *q* e e nfig r i n *R* se r g s e is gi en in ig re s n n e p r i i n n ere ig e is si i r nfig r i n ing n 6 n e p r i i n

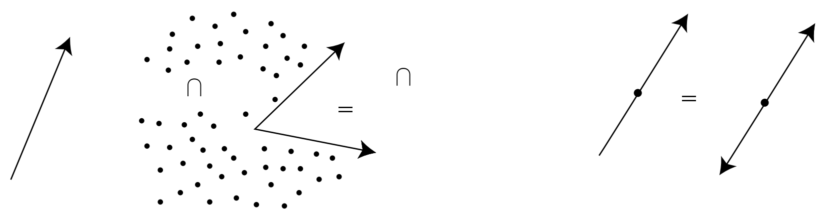


g c ra a 8 8 4) ala c ar

OO O O

n is se i n e s gi e s e p r s pr s e re s s e en i ne e re e e n i *d r d l s* in r er efine e rig si e ine n e e si e i s *l* e ns ire e ine A ine isse s e p ne in ree pie es: n pen p nes R n ere R n en e e *p h l pl s* i re n e rig si e n n e e si e respe i e see ig re e n e r s e n ing r e s e p in en e en e R e pen regi n is s ep e r eing r e ise r n r n es n n in e p in see ig re i i r e efine e pen regi n R i is s ep e r eing r e n er ise r n r n es n n in e p in en isse s e p ne in ree pie es: n pen regi ns R n R

e in ern ng e _ _ R is ess n en e R e *w d* efine n n en e i r



g O r s)) a $_1) \cap$ $_2)$ a a w *w* $_1$ $_2) =$ $_1) \cap$ $_2)$

e e ine i s n p in n e efine s e ine ing n n ing e pp si e ire i n e ne efine e r s n ing n e ine n ing e s e s r ing p in s s e s e ire i n s n s e pp si e ire i n n p r i r see ig re C n erse gi en r e n si i r efine e r se ire i n is pp si e n e ine i s e s e ire i n s

r ine i p sses r g s e p in s in R ere e is ines n ine r er s r ns i ns p ss r g n p in s in R n s is R R n R R R R e en se is i en i ning

H nd ic r) *L R d d s s s r d p s d lu p s h pl r sp ly h h r s s l su h h R R R R* $\leq$ *R d* $\leq$

I R d h s p ss s hr u h r d p d lu
p h w s y h R s p r d w l d su s s y

e ing e is n n n i s pr n e n in n

L R d d s s s r d p s d lu p s h pl
r sp ly I h r s w l s d su h h R R
d h y h p ss hr u h s p s R
h r ry r ≤ ≤ h r s s l su h
h R R d p ss s hr u h p
R

e ne e n e pr e e s e rg en s in e pr
e e e is e n n in s e ine r in
s *R* *R* nges n p sses
r g e ne re p in

L R d d s s s r d p s d lu p s h pl
r sp ly I h r s w l s d su h h R R
d h d p ss hr u h ly r d p
r sp ly d h p ss hr u h lu p s h r ry r
≤ ≤ h r s s l su h h R R
d p ss s hr u h ly r d p d lu p

L R d d s s s r d p s d lu p s h
pl r sp ly I ≤ R ≤ d h r s s l su h h R
d ≤ h h r s s l su h h R
d p ss s hr u h p R

r pp se firs *R* e e ine p sses r g e ne re
p in n s isfies *R* *R* *R* en e s ne
n *R* n ins e s e p in s pp ing e
n ei er r e n in e esire ine
e ne nsi er e se *R* e e ine p sses r g
re p in s n s isfies *R* *R* en e s ne
n *R* n ins e s e p in s n s e e s
e

L R d d s s s r d p s d lu p s
r sp ly I h r s s l su h h R d ≤
h h r s s l su h h R d p ss s
hr u h p R

r C nsi er ine p ssing r g e ne re p in n s is ing
R n *R* *R* en r *R*
s e ere e is s e esire ine

r iti n *L R d d s s s r d p s d lu p s*
h pl r sp ly h R h s l d p r

r n ess er ise s e e ep en i es e s nsi er ine p sses r g n p in in R e egin i C i s

i ere e is s ine s R n
$\leq$ en e pr p si i n s s e ss e r e er ine i
R i s nsi ering e s
ss e r e er ine i R R R i s
R

pp se ere e is s ine s R n
$\leq$ ere e is s ine s R n en
ere e is s ine s is ing R n e
s n i e re R R n e p r i i ne in n e
s se s s e pr p si i n s ere re e ss e r e er
ine i R i s i i r e ss e
r e er ine i R R i s R
e n e p r e ines i e s e ire i n p ss r g
e ne re p in respe i e n s is R n R R
en R n R n ins e ne re
p in e s e r e ss e ere e is s ine is p r e
ies e een n n s isfies R n
e n R R s n e p r i i n en e
e pr p si i n s C nse en C i is pr e

i ere e is s ine s R n
en e pr p si i n s s e ss e r e er ine i
R i s

pp se ere e is s ine s R n
en ere e is s ine in R s p sses r g e ne
re p in n n e p in R R R n
R $\leq$ pp ing e n ere e is s
ine p sses r g e ne re p in s x n n e p in n
s isfies R n
e n nsi er R x i n ins ree re p in s
n e p in s e R x s n e
p r i i n n s R s e esire n e p r i i n ere re
C i is pr e

e e ine p sses r g re p in s s x n n s isfies
R see ig re R n R R
nsi ering ine er se s R x R n
i s r C i
C i e e R i i p ies
ere re

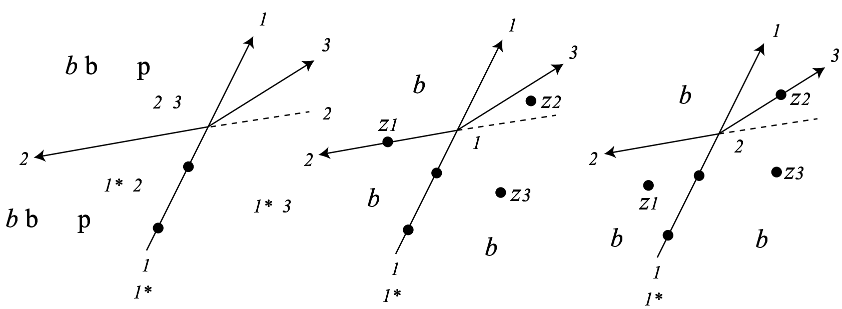
n R

e e n p in n n r e n ing r s R
n ins e e p in s ere en es e r n e n ing

r n ing e pp si e ire i n is n ine in n s R is e ge e ne e r e n ing r s R n ins e e p in s is n ine in R n i R n ins re p in en s e n ine in R C i n s R is e ge

e firs nsi er e se x is e e p in n x n is se R is e ge n ins n re p in en is n ine in n e ge e esire n e p r i i n x n ere n ins e re p in s n en e e ss e n ins e s ne re p in en x

ing p in ng in i s ire i n r x p in er r r x e n fin ei er i p in r i p sses r g e re p in n n ins n re p in s r ii p in r i p sses r g ne re p in s n n ins n re p in see ig re in e n ins e e p in s in e se e n e si in e esire n e p r i i n



g l 1 a ra s 1 1 2 a

r iti n *L R d d s s s r d p s d* 6 *lu p s r sp ly h R h s l d p r*

r n ess er ise s e e s nsi er ine p sses r g n p in in R e egin i s e C i s

i ere e is s ine s R n en e pr p si i n s s e ss e r e er ine i R i s nsi ering e s ss e r e er ine i R R i s R

pp se ere e is s ine s R n s i e er s r i n e ss e e er ine p r e p sses r g s ne p in in R en ere e is ines n in R s e re p r e n e e s e ire i n s R R R R $\leq$ n s e n p sses r g e ne re p in in p r i r R n ins n re p in s

pp ing e n ere e is s ine p sses r g e ne re p in s x n s isfies R n en R n ins ree re p in s n e p in s n R x s n ins ree re p in s n e p in s e s n e p r i i n en e R s e esire n e p r i i n ere re e i is pr e

i ere e is s ine s R n $\leq$ en e pr p si i n s s e ss e r e er ine i R i s

pp se ere e is s ine s R n $\leq$ e n ss e ere e is s ine s R n en e ere e is s ine s R n nsi ering er s r i n i ne ess r e ss e e er ine p r e p sses r g s ne p in in R

e e ine is p r e p ss r g e ne re p in n s is R en C i R n ins s e p in s i i p ies ine er se n ing e rig s isfies R R n R pp ing e R R n e in R R s n e p r i i n i i p ies R s e esire n e p r i i n

en e e ss e r e er ine i R i s

s n i e re R n e p r i i ne in n e s se s e i n ins e ree re p in s n e p in s e n e e s e en n ine e s se s n e p r i i n en e R s e esire n e p r i i n ere re e i is pr e

i ere e is s ine s R n $\leq$ en e pr p si i n s s e ss e r e er ine i R i s

pp se ere e is s ine s R n $\leq$ nsi ering ine p ssing r g re p in s n s is ing R R R e ss e s e r s ere e is s ine s R n e e ine is p r e p sses r g e ne re p in n s isfies R en C i e n er e p in s ing e een n is s en e e R R s n e p r i i n s R s n e p r i i n n en e e i is pr e

e e ine p sses r g re p in s s x n n s isfies R R n R R nsi ering ine er se s R x n i s r C i s n is e ing ine i s:

n R

e e n p in n n r e n ing r s R n ins e e p in s ere en es e r n e n ing r n ing e pp si e ire i n is n ine in n s R is e ge e ne e r e n ing r s R n ins e e p in s is n ine in R n i R n ins re p in en s e n ine in R C i n s R is e ge ere er si i r rg en s in e pr r p si i n e n pr e r p si i n

e i e pr e ing r p si i n in e se r p si i n e n in i i pr e i si i r rg en s in e pr s e e pr p si i ns

r iti n *L R d w d s s s r d p s d lu p s r sp ly I h R h s h l d p r d l d p r I h R h s h l d p r d l d p r*

e re n e e si pr e sing r p si i ns n s n i e re n e re r e p e e e is en e n e p r i i n is g r n ee e re n n e p r i i n is ine r n e p r i i n pp ing s n i e re i s s se n ining re p in s n e p in s

e n gi e s e e pr e re sin e i is i e ng i is si i r e pr gi en in

e R n e is in se s re p in s n e p in s respe i e e pr e e e re in i n n

i en R s n e p r i i n s e ss e

i ere e is s ine s $\leq$ R $\leq$ n en R s n e p r i i n

e gi e pr C i sin e i is s r e in eger efine in e i is en R s n e p r i i n e in i n p esis in e is e en R R s n e p r i i n s n i e re en e R s e esire n e p r i i n is e en en e pp e in i e p esis R R n e n si i r in e esire p r i i n s e

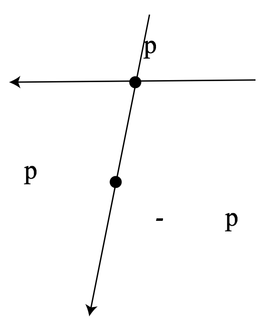
i e e n in eger s $\leq$ $\leq$ n en r e er ine i R e ss e sin e er ise e e re s re e i R en e ss e

e e ine p ssing r g ne re p in s x s R n ins n re p in n e ine p sses r g x n ne re re p in s s R R n R i ss gener i e ss e e ire i n is n r e e r e n ing r x s R n ins e e p in s

i e ss e R n ins e s re p in s sin e er ise e e re s

e e r e n ing r x s R n ins s n s p in s in R s e R n ins e re p in s n s e p in s n ies in R

e e is en e e e is g r n ee C i pp ing si i r rg en s gi en in ese ines n r s n n er r s e n ing r x e n pr e e re



g w l s 1 a 2

1 k a a k a aka ra E ra a S k a a a rr a a cal r c ar s c s s la *l* c r s c r sc c 17) 1 1)

S s a a k rk a r ck a S k ral z a sa w c c s q a l s s s r r
a a O rk *b* *l* r ss 11 1997)
4 ara a k a a al a sa w c r r ar r c c s *l* c r s c r sc c 17) 1 9 1 7)
a k a a ala c ar s w s s s la a al r r a l ca s **1** 1 1999)
a k a a ala c ar s la a r r l s r r
a al r r a l ca s **1** 1 1999)
7 a k a a r c ar s c s s la r r
8 Saka ala c ar s as r s 2 a ar *b*

P hs mp p m c ph

A s s i K ne n K n

[1] ar r Sc c a ca E r ak rs S k k k 1 8 77 a a
[2] ar r a r a Sc c s arak rs ac 1 8 11 a a

bs ac A a B w s s s s la s c a r s A B ar c ll ar a l n r s A r c c l ar ra A B) s a c l ar ra w ar s s A a B w c s raw la s c a ac A B) s a s ra l s W r a) B n 1) n 4) 1 r c c l ar ra A B) c a s a a a ass s r all s A a as cr ss s a) r s s a c ra A B w B $= \frac{n}{16}$ $\frac{n}{2}$ 1 s c a A B) r a c a s A as a l as cr ss

O O

e e fini e gr p i ps r ip e e ges e en e n e se er i es n e se e ges respe i e r er e v e en e eg v e egree v in r se e en e e r in i *A r r ph* is gr p r n in e p ne s is se p in s in e p ne n ree i re ine r n is se p ssi r ssing s r ig ine seg en s se en p in s e ng ge e ri gr p is p e e ip r i e gr p i p r i e ses n i e en is en e i e e *r pl p r r ph* n 6 A e n s J r i ern n e n s s e e ing res

r n t) *L d w d s s s p s h pl su h h d hr p s r ll r h h r pl p r r ph s sp r w h u r ss s su h h h u d r s* g

n K ne i pr e eir res n pr e e ing e re

r K n) *L d w d s s s p s h pl su h h d hr p s r ll r h h r pl p r r ph s sp r w h u r ss s su h h h u d r s s*

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 187-191, 2001.

is e n n n er e s e n i i n in e re ere re
nfig r i ns s es n n in i ni n p
i r ssings e e pper n e n er r ssings
i ni n es in is gi en in e re e e ing
pr e i en is in se s n p in s in e p ne s n
ree p in s re ine r i is rge p re i en es
n in p i r ssings s n ins e se
e ns er e e es i n is in e r i e s e s see n e
pr e e ing e re

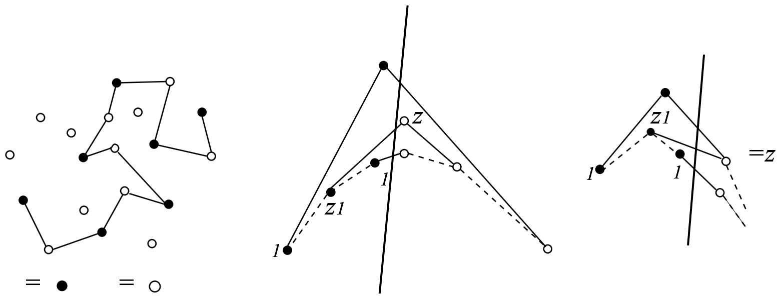
r *L d w d s s s p s h pl su h h*
hr p s r ll r d l h u r p s
) I h h r pl p r r ph
s p h w h u r ss s su h h s h s
) h r s s ur w h — – su h h
ry p h h s h s l s r ss

n r er pr e e re e nee s e n i n n efini i ns r se
p in s in e p ne e en e n e n e e n r
n is p g n se seg en s n e re es re e *h d s d*
h r s n respe i e r p in s x n in e p ne e
en e x e s r ig ine seg en ining x i e n e ge
ge e ri gr p n ining x n s i er i es e e se
p in in e p ne e e er e n n e x e p in e eri r
n en e s *x s s* n n i e ine seg en x in erse s
n n

L R d S d s s s p s h pl w h R S
su h h hr p s R S r ll r upp s h h r s s l
h pl h s p r s R d S L x d w r s n *R S*
su h h x S R d x s d n *R S h R S*
h r s s p h w h u r ss s su h h
) h r x s d d
) p ss s hr u h ll h p s

r e pr e e e in i n n *R S S* r *S* en
e e s i e i e n s e ss e *R S*
e x e e er e n *R S* s x *S* n xx is n e ge
n *R S* see ig re
en e n fin p in s *S x* n *R* s x n see
n n is n e ge n *R S x* see ig re e
i r x n r i i r e n fin re
p in s *S x* n *R* s n see n n
is n e ge n *R S x* see ig re e i r
n r i
e n pp e in i e p esis *S x R* n en
ere e is s p in *S x R* i r ssings s r s i

n n ins S x ing e ges n x e in e esire p in e e



g 1 a) a r 1) r r a 1

e pr ee pr e p r i e re e ss e n p in s e e s e x r in e e e p in s S s r e eir x r in e n e e e er i ine i p sses r g e p in $\leq$ $\leq$ ese ines sep r e e p ne in regi ns n en e e sep r e e se in is in s se s Ass e ese ines re ire e p r e ss p i n e s ne s se n ins e s p in s e ss e ne e regi ns i n ins e s p in s is n e e ines n $\leq$ $\leq$ e e s n rig s n n e regi ns n e re e si i r e e e s se e een n i e e e e ine e een n s is ing e ing n i i ns:

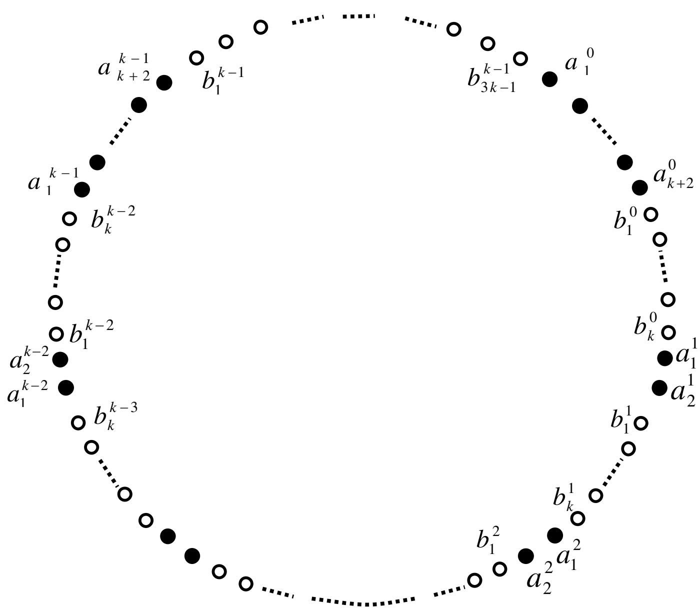
i p sses r g p in n is ire e p r
ii e n er p in s in e e is

e l e e s se e e n l en l n e l e e s se e e n e e e s se e rig ri i l n e n e e r s e n ing r s is ngen n l $\leq$ $\leq$ n is e A s e n e e r s e n ing r s is ngen n $\leq$ $\leq$ n is e i e sin e n ree p in s re ine r e r n ins n p in l e l e e s se l e e r n l e s se l n er e r A s e e e s se e e r n e s se n er e r in e l e e ei er l r l s l i i r e e ei er r s C nsi er n l l in e l l pp ing e n e ing

x e n fin p R_l in $_l$ $_l$ i r ssings s i e er e is n en R_l n ii R_l n ins $_l$ n si i r nner e n fin p R in i r ssings s i e er e is n en R n ii R n ins e R_l R C e r is p in i r ssings s n ins e se

n r er s p r ii e re s pp se n p in s n ie n e in e ing r er:

see ig re



g

is n i s n — – n in e er p n ining e se s e s ne r ssing

is p e es e pr e re

L D

e i e n r ess r n r s ing s e nfig r i n

1 lla as arc a r á z a a s
ar s r s la *g 6* S r r rla
S **11 0** 199) 1 1
k a a a rr a S l al r a a r l scr a a cs
1 199) 1 1 1
a k O a r ar s r s
la *l* S r r rla S **1**)
1 171
4 a k a a s l r a a l c cl s w
r cr ss s la r a al r al a al r
& l ca s **10** 1) 7 78

pp m g m T g M sh s ph s

i K t Hir ic i K i y ni c i

[1] ar rc c r a rc c ral S s s rs
s a ac Sak k 8 1 a a
na a c ac , ma m a c ac
[2] a lwa r a S s s
kar ac k k 18 8 1
_ an c c

bs ac W c s r r l r a la a c l s r s s n S r s a z s rall l ra W s a l s a r la s r l a c r a r ack r l as s r la s w l a r s c r c a r a r s r s r n s c s s c l lar) a s r c r a lar s s s r s c s as l a r s n) a n) s ac

O O

i en n e p g n $\boldsymbol{P}$ n e sp ere see e i n r e pre ise efini i n r n e p e r n se in is p per n p si i e in eger e nsi er e pr e gener ing eng ni r ri ng r es r $\boldsymbol{P}$ sing einer p in s re spe ifi e n fin se S p in s in e s re $\boldsymbol{P}$ n ri ng i n $\boldsymbol{P}$ sing S ini i es e r i e i e ge eng e ini ne

n r re en r e nsi ere e se n e p g n in e p ne n pr p se 6 ppr i i n g ri r is pr e n is p per e e en e res sp eres

A g e r ness s n een se e e fin ing n p i s i n r e pr e r e n e p e r n n e p ne r e sp ere see s e i in ie e p e eness p ing pr e s in e p ne see e g J ns n r in ie e in rinsi p e i ei r nn's ri ng e pr e see

n r er e p n ers n r pr e i e n e r nsi er ne i ensi n n g e pr e e gi en n in er n ir e i spe ifie p in s n i p e p in s in e in er s s ini i e e r i i n ini in erp in is n es n e n r r e in rinsi i e pr e e i in is p per ne i ensi n pr e n e e s e in s r ig r r nner g e e i s re i e ere A ern i e n i gine si p e gree pr e re i er i e

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 192-204, 2001.

p es ne p in e i e e nse i e p in s re r es p r e $_i$ en e e ini in erp in is n e r e rigin p in se A er e ini in erp in is n e e es s er n $_i$ e r i e es ess n A s s g es infini e r i n erges ne e re gi en ine seg en ins e n in er ir e e pr e essen i re ins e s e

en ing is i e i ensi n se e e pe 6 ppr i i n g ri r e se n e p e r n in e p ne e g ri ses e ris i e *l r s r* i ppr i e s es er in e re e p ing pr e r p in se s i in $\boldsymbol{P}$ i e e pr e s s r p ing ir es i en ers in $\boldsymbol{P}$ s e s es r i s is i s een s n is e ris i gi es ppr i i n g ri r is pr e n e De n ri ng i n r e p in se ppr pri e ifie r e s i n pr e e g ri gi es 6 ppr i i n

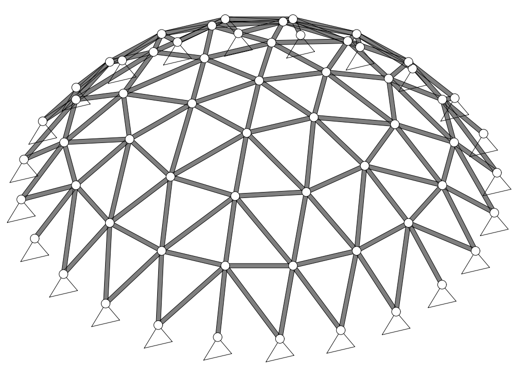
e g ri pr p se in is p per is se n e s e i e s e ne e e pe in e se g in n ni r n i inser i n r e sp eri regi n pr e p in se n p s n ppr i e s i n e n e r e p in se ppr pri e ifie r e s i n pr e e n ni r n i inser i n e r ifferen e e een e g ri gi en n e ne pr p se in is p per ies in e nee rr D r n i i gr p i n r i p e en ing n ni r n i inser i n n in r er es is e ppr i i n res e nee n n ri i rg en s in r er gener i e se er e s gi en in sp eri s r es

ere n er e se D r n i i gr ins e D n erp r r sp eri regi n A e se D r n i inser i n g ri r sp eri regi n e nee se e e ri se n e ge esi is n e e sp ere A er ining e De n ri ng i n e sp eri regi n e e ge is n s r ig ine seg en in r r e gre ir e p ssing r g e en p in s r e e ge is en rep e s r ig ine seg en in e fin s i n n is se e er s s n nig i e rs se r i i n ini e ge eng s e es i is rger n 6 n e er n e s s se n D r n i i gr e r i e i n ini e ge eng s is s 6 i es e n r e p n r se re e i e s n e r i n e i pr e ϵ n er er in res ri e n i i ns e er r nning i e is

e e rrie p i n e peri en s see e effe i eness e pr p se e e p i n res s s r pr e ins n es re es e e r i i n ini e ge eng s is s s ere is ig g p e een e re i n e peri en res s se e g p re ins s re rese r

r s gener ing ni r ri ng r es es n sp eres is i e e ne essi esigning s r res s s ri ng r r sses rge sp n s r res see ig ere i is re ire e er ine e s pe r es e i p in s ie n er e ns r in s n erning s ress n n isp e en e s fin ing n p i s pe sp e s r res s een s ie se er rese r ers e r ss n e ie e s ri ng i n p in s n e r e s r e reg r ing r ss e ers n n es s e ges n p in s respe i e en sing n e s pe e ge eng s s e s e s p ssi e r e ie p in esign e ni s n n ring n s pp i i ns e i ns e p in s re s n fi e n e ie e s e isi n ri es n ie is i is i e n r nsi er e pr e e p per s in e s ie e s e pr e s rs n pr p se e ris i ppr se n p en i n i n efine r e er p ir p in s e p en i is pr p r i n e in erse s re is n e e e fin s nfig r i n p in s ini i ing e p en i e ris i e e ppr e e pe erein is s e i pr e en e

r e pr i ie p in s se er er rs e e en in nsi er i n r ins n e i e n er is in e ge eng s s e s s s p ssi e e se is i e re se e pr i n s e ers ii e ri ng r r ss e esigne s p ssess er in s e r pr per i e ini i esign s s pr per n e s r rese r n e ie e s n en ne in ie e pr i esign p i ri ng r r sses



g 1 ll s ra a r a lar r ss

e ing n i n i e se r g r p in s x n n e sp ere e x en e eir i e n is n e e ini n n er is n e e een p in se s n is efine s in x x x en is sing e n se x e si p ri e s x e efines e ini in erp in is n e ng e p in se

o ooo o x k

A regi n n sp ere is i r n p in s n v in regi n e s r er r nne ing n v n gre ir e p ssing r g n v is insi e e regi n A n e regi n e re n erne i is ss e e n ine in e isp ere; re spe ifi n pper e isp ere i e n r e isp ere ge er i n e regi n n sp ere se p in s e er i es n e n r e regi n is gi en s n inp n e regi n ge er i n r p in s is e *p ly* n sp ere i e e n r e ge n e p e r n efine ere is n ine seg en in D sp e r e seg en r gi en p in n sp ere n p si i e n er *sph r l d s* en ere i r i s n sp ere is regi n e sp ere s r n p in x in e regi n e i e n is n e r e p in $\bar{x}$ is s r r p rp se is ss e e s er n r e i es e r i s gi en sp ere

n is se i n e nsi er e ing *r p pr l* e $\boldsymbol{P}$ e se n e p g n i er e se

i i e S S
s e se S p in s i in $\boldsymbol{P}$

n er r s e pr e s s r p ing is s i en ers in $\boldsymbol{P}$ s e s es r i s is i e s gi e ppr i i n g ri r is pr e sing *l r s r* n e i n e en s e p in se S pr e is g ri s e s e ri ng i n in e e pper S n e ifie gi e n ppr i e s i n r e pr e resse in e i n e g ri gi en ere is s r ig r r e ensi n e ne pr p se in e sp ere n s e i e pr e e n e e re gi en e

e g ri e er ines e i n e p in se S in gree nner e s r ing i n e p se S i repe e p es ne p in insi e $\boldsymbol{P}$ e p si i n i is r es r e se S e i e e g ri rigin es i n e n e er n reene n s e e pe r ppr i ing ini s erings C p r e inser i n s r egies re s se r es gener i n in D in C e n in pper ere e *l u y r* eir s r egies i ifferen i e s res e er n inser i n es n e p e in n ni nner ri s res s n e si e ir e p ings re s ri e in e es

e g ri is r es ri e e ses e r n i i gr e rren p in se se e e ne p in e inser e e ss e i i ri i e si pr per ies r n i i gr n i s e De n ri ng i n n re er e s r e p per n e ing g ri e se D r n i i gr nsis ing r n i regi ns A r n i regi n is n e p e r n in

rit

t ni i i e S :

t C p e e r n i i gr in D r S S

t in e se in erse i n p in s e een e ges r S n $\boldsymbol{P}$ in e se C in erse i n p in s e een es r S n e n r $\boldsymbol{P}$ A ng e p in s in C se e p in i i i es S

t S : S n re rn ep i S

e n S respe i e en e e p in sen in ep n e se ine in ep e i er i n e g ri r n r i r r p in x $\boldsymbol{P}$ efine e w h x i respe S s x x S is x is e r i s e rges is en ere x i es n en se n p in r S efini i n r n i i gr e p in i i es x er x $\boldsymbol{P}$ e

$$S \quad S$$

e e ini in erp in is n e re i e S r er re en e S e p i s i n r e e re e p ing pr e r $\boldsymbol{P}$ n e en e e rresp n ing e i e e s pr is n p i n e ni es in n n ins ser i ns i e se in r r er n sis

r *h s lu S d y Al r h IN s ppr h r p pr l r* $\boldsymbol{P}$ *h s*

r e pr is ne pr ing e i re i es e ini n n er is n e r S S i en e i is

$$S \quad S$$

e re ining p r e pr is gi en in

r y s S $\boldsymbol{P}$ *p s h r s s p x* $\boldsymbol{P}$ *w h x S*

e e ss e p n e er e se $\boldsymbol{P}$ is gi en s n inp esi es s er e se e p in s insi e e p g n re s e i es ini i gi en i e e e e re n e s r is se e n e e r nning i e e g ri e in ing p r is e p i n D r n i i gr i re ires i e in e e e p e D r n i i gr n i es e i e is

y o ox o o

r e p n r se e g ri p es e De n ri ng i n p in se s e fin p n e er n r se S p in s n $\boldsymbol{P}$ e i p e pper p r e n e e se s i s ri ng i n

re **P** is n ine in e pper e isp ere in e e n e
S is p r e De n ri ng i n D in S e s
en e e pper p r e n e S D S
r $\leq$ $\leq$ nsi er e ri ng i n D S r ri ng e
D S e en e e s r ig ine i is e s e p in s i
e is n e r ree er i es in erse s **P** is e *r* *l*
er ise *r* *l*

N d e r l r l S) s l r h

r e *e* *q* n en e i x e in erse i n p in n **P** As
x ies insi e **P** e ge x x q x e i e p in
in ep A g ri e e x $\leq$ e ri ng e
ine i n i p ies q $\leq$

r ne ser i n is n ri i ri ng es C nsi er s e e ge *e* D S
se en p in s re n e n r **P** ge *e* s ffs e p r
e r n i i gr r S ies si e **P** p r n ins
r n i er i es en e efine e *r l r R e* r *e* s e ni n
e ri i ri ng es re ese er i es As ser e in i
is n r see e ri i ri ng e D S e ngs ni e
ri i regi n

N d r l r l R e s l r h e

r e e n en p in en e r n i regi n in r S
in erse s *e* e n se p in x n e sp ere i is insi e is regi n
si e **P** ere is s e en ere x i en ses n en p in
e e ge *e* s ff p r i x **P** is p e e ere
e e *e* i i e er *e* is i p ies ies in *e* ere re
ere e is s s e en ere x i en ses n en p in *e* s
is p e e ies in e *e* i i e er *e* s ies in *e*
i i r e er en p in q ies in *e* nsi ering en ere
x i en ses q As e is n e e een n p in s in *e* is s
e e ge $\leq$ *e*

e s r er is ing is e een *d p d* e ges D S n *d p*
d nes e er pe ing en p in s in e eng in epen en
e ges n e n e s s

h d p d d e S) h s l h l s

r e e *e* S S e se *e* s s ne en p in in
r e n S S

e ren re s ri ng i n i e ge eng s re e n e p e irs A g ri is r n n $\boldsymbol{P}$ in r er p e e e e ss e n is sen s ien rge ss re $\leq$ is ss p i n is n nn r s e s r es e ge e esire ri ng i n nn e nger n e s r es e ge $\boldsymbol{P}$ A er ing i e p in s re p e n e n r $\boldsymbol{P}$ i nse i e is n es e een n n s s r i e s p e en is s p ssi e e pr is i e ere en e inner ng e $\_$ r ree nse i e er i es is rger n r e 6 e n p e p in s n e n r s e nse i e is n e is e een n n s in i i n p in s re pr e re r nning A g ri er is p e en

r $\leq$ $\leq$ e S Define x x S r p in x $\boldsymbol{P}$ e e i rn e r i r n ing e e ge eng e i r e ri ng i n D S e 6 e sser s ge s s en e ee s ns n ip e e n er pres ri e p in s

h r r d y sph r l d s w h r d us s u l

r pp se e en er e is is e n r p e s N e en er e efining e sp ere n n r i r r p in n e n r e is re en e n respe i e e R en e e r i s e n e en e e ng e $\_N$ en s R e re e is is p e

$$\int \quad R \quad R \sin$$

is e i e se in e pr e ing e

6 *r y p s s supp s* —— *h* $\leq$

r e p in se S pr e A g ri in e firs r n is rge en g ens re e ge r As p in i i es x r x $\boldsymbol{P}$ e sp eri is s en ere e p in s in S n i r ii p e e er e p g n $\boldsymbol{P}$ e r e en e en e r e

$$\boldsymbol{P} \leq$$

ere $\boldsymbol{P}$ is e re $\boldsymbol{P}$ n en es e re si e $\boldsymbol{P}$ i is ere e sp eri is s en ere

Ass e n Dr sp eri is s i r i s – r n e p in in S S in e S S S ese is s re p ir ise is in e s ere s n n e se n r is n es efine

S re s ese is s ie p e e insi e $\boldsymbol{P}$ i s ese is s re s is in r e is s r i s en ere in e er is s re p ir ise is in sin e $\leq$ C nse en

$$\boldsymbol{P} \quad —$$

ere en es e re insi e $\boldsymbol{P}$ i is ere e is s en ere C ining n n ser ing n i p ies —— n r i i n

pi en e e s s $\leq$ i s e ser e e n er epen s n e ing g r n ees e ss p i n in e 6 pr i e is s ien rge e $\boldsymbol{P}$ en e e peri e er $\boldsymbol{P}$

$$h \quad d \quad \leq \quad \boldsymbol{P} \quad \boldsymbol{P} \quad pl \quad s \quad ——$$

r e e

$$\frac{\boldsymbol{P}}{\quad}$$

ge n n ser e s e p in s re p e n e e ge e $\boldsymbol{P}$ is s s p

$$\leq \frac{\boldsymbol{P}}{\quad}$$

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e ing is in e re is p per

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r e pr is essen i e s e s e ne gi en in r e p e eness e pr is gi en ses re is ing is e r ing e e

C se : C n erning pper n s e i p ies $e \leq$ r e ges e e nging n n ri i ri ng es e e ngs s e ri i ri ng e e s s e nn e rger n e i e ge eng n e n r $\boldsymbol{P}$ i is s ns r i n C n erning er n s e gi es e r in epen en e ges n epen en e is e n i respe n e n r e i i p ies e e se r epen en e ges is e ges sp nne e ri i in e s ns r i n

C se : e pper n r n n ri i ri ng es n gi es $e \leq 6$ e e s 6 n i e er n r in epen en e ges e es e e re ining n s re e s e s in e r er se

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en e inner ng e _ r ree nse i e er i es is rger n r e 6 e n p e p in s n e n r s e nse i e is n e is e een n in r er s is n is se e n ge n i is e er n 6 i is rge en g s —— s is res s s r e p n r se n s s reng ens e ne gi en

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e e i p e en e e pr p se g ri e s s e peri en res s ine pp ing e g ri r e e isp ere i r ini i p in s n e n r s s n in ig e e es e e ses e r i e nges n s r es e ge eng s r e se is gi en in e n e e r e e es r i n

Tab 1 E l ra s ra a l

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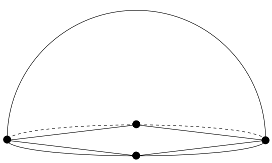
ra	1 9	1 9 1	1 9 4	1 9 41	1 9	1 9 7

r i represen s e r i ine e firs r n n e ne e se n r n r n i inser i n g ri respe i e is ser e r e e e e ge eng r i is se i is e er n e rs se r i r ≤ ≤ gi en C r r n r e peri en s is rger n pi is r n r e e isp ere es e in r e peri en s e n p e p in s n e n r s e is n es nse i e p in s re s e s e i is se in r e per i en s e fig res e s e s i ns ine e g ri r



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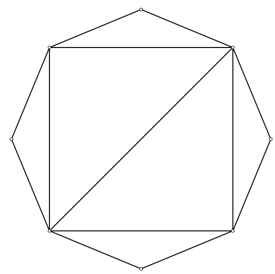

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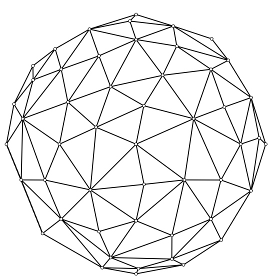

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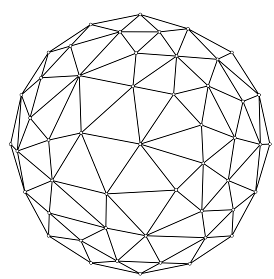

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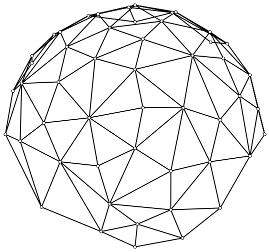

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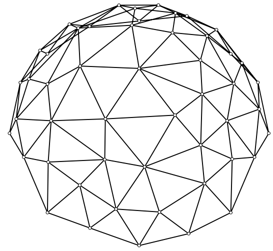

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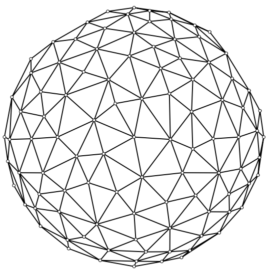

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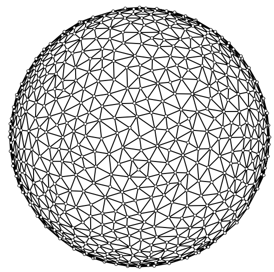

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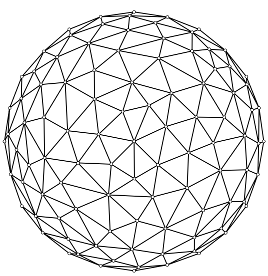

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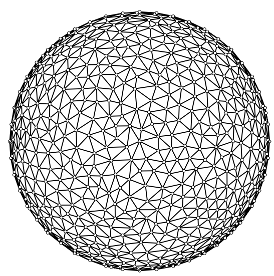

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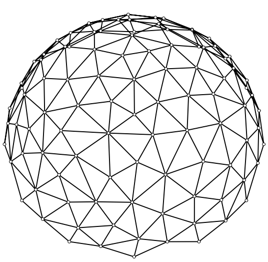

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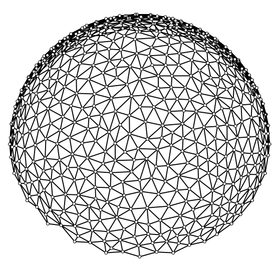

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eng r i s s e g ri firs p es e p in s n e n r s nse i e is n es re e een _ n β_ n _ s β is sen s s s p ssi e is s ere is e ne efine in e i n App ing e r n i inser i n g ri e en p e e p in s in ***P*** s n s p ssi e s ng s e ini in erp in is n e is e s _ n e n r r s es ri e in e pre i s se i n e e e e e r n i inser i n n n e en e e e ing e re

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is rese r s p r i s pp r e e r n in Ai r ien ifi ese r n ri ri Are s e inis r i n ien e p r s n C re J p n e rs i e n e n n s re eree r e p en s

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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 205-216, 2001.

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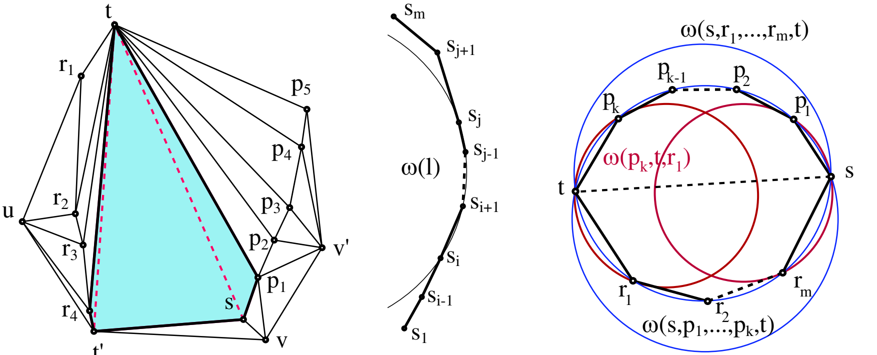

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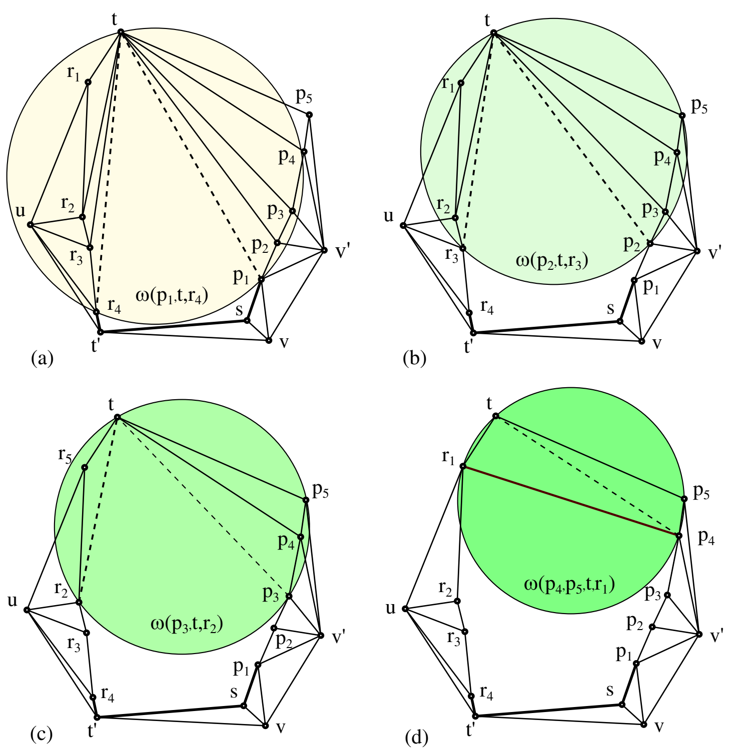

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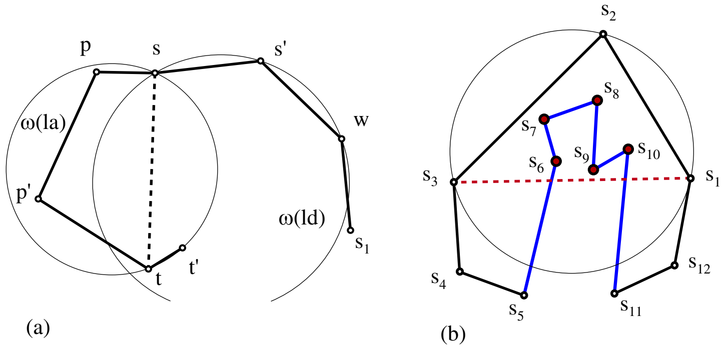

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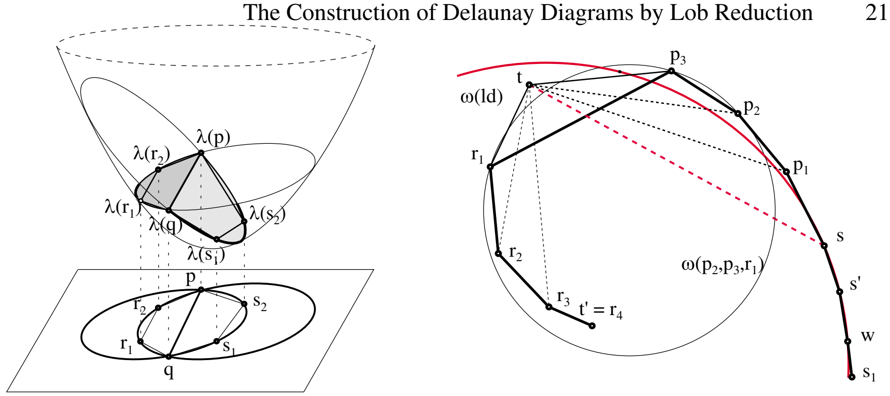

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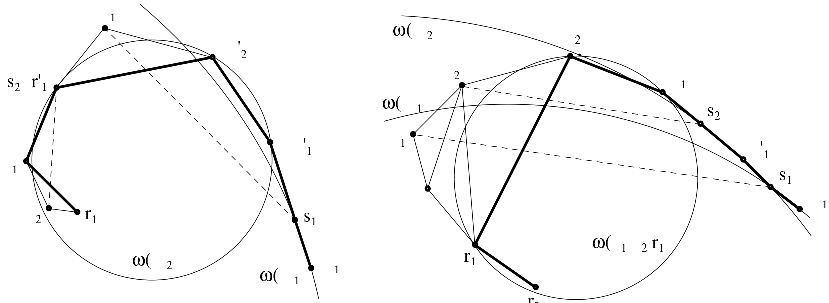

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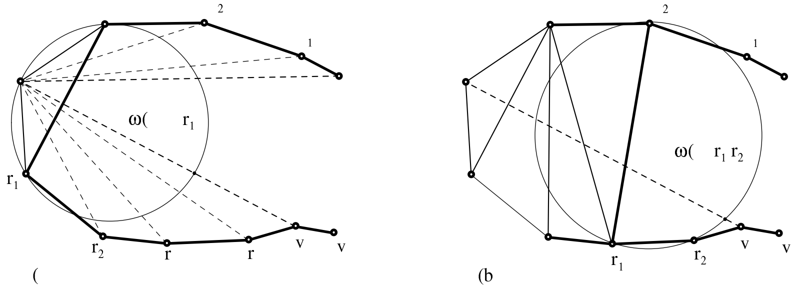

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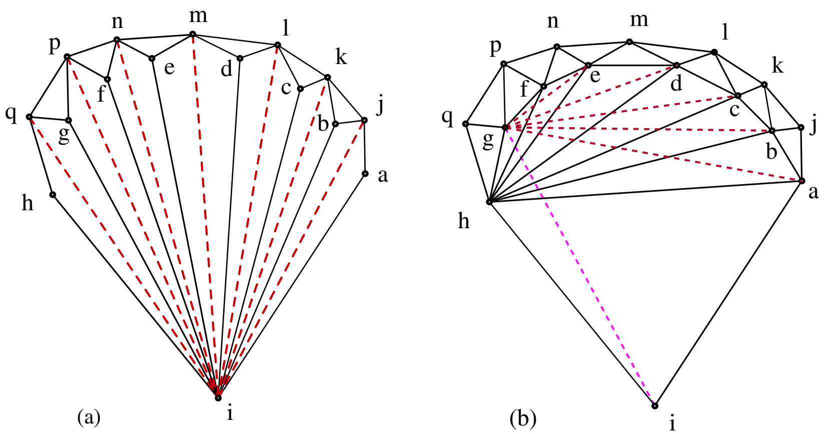

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B B i F r r bu r i r c i
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8 L r i u ri u i *l* *t* *a*, –108,
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t *S* *C* *put* , 22 –26 r Sci i c, Si r , 2
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t *C* *put* , 22 333–3 6, 1
u u *La* *t u t* *u* *a* *a* *D lau* *a* *pa* *ala* *a*
a *l* *pla* *t* *appl* *at* i , U iv r i H u –A c ,
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L L L S r r 1 ur c i r i ic , i r,
at *S* *t* *a* , 161–1 Ac ic r , N r , N , 1
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SH M S H i r b *P* *t* *u*
S *p* *u* *C* *put* *S* , 1 1–162, 1
SS Sc i S r A u r r r i u i r i
i i *D* *t* *C* *put* , 1 –36, 1

eomet T somtos l e T ul tos

K n ic i K r s i , s ir m ,
Y s i i Od , nd M m r n

n o s,
o n s y, o o 22 8 22, J n
n o s,
Os yo n s y, s 8 2 8 82, J n
3 n o s nd Co n ,
n n s y, s 90 8 0 , J n
n o Co n nd s,
s n s yo n s nd s, s 12 8 0 , J n

b a . n s , s n s l g o ns o ons, so s ll d on ons n g o y, n l n ng l ons. os ns o ons n b l d o s l o l z ons o g o o s x. n b o ngl s n l n ng l ons s n y s d only o lo l g on so dj n ngl s . W o s n l on sl g ly.

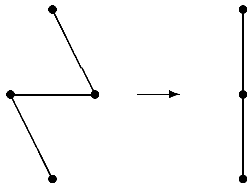
c i

c nsid r n *s m* r p s, is, r p s i n ps nd n m ip d s m dd d in p n s n d s d n cr ss c p ir nds p n *g* is p n r p i c c ri n r F r p n r p , d n i s r s , d s nd c s V , nd , r sp c i d n minim m d r r p δ - m ns c c n s $\subset V$ is - i | | = nd r p is disc nn c d r ic s in minim c in p n ri n i n i n c mm n c c C n is c s , s C is *s* *g* c c nd c d C is in c in ri r C is d n d n C nd s r ic s in n C is d n d V n C

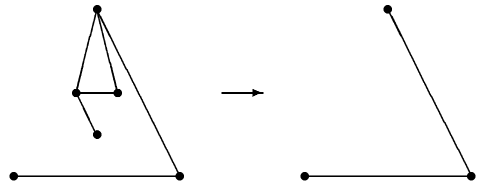
p n ri n i n nd n d *g* r in is r m , id n i nds nd r p c p irs m ip d s sin d s r sp c i , s s n in Fi is n n r p n ri n i n c n r ns rm d in r dr n s q nc c n r c i ns d s [

F r p n ri n i ns i minim m d r , in r m is n n *m* *f* is n p r i n s n in Fi 2

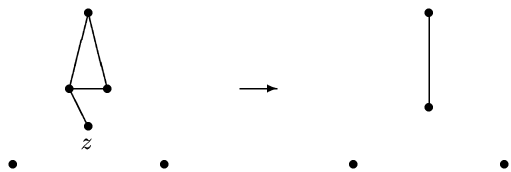
J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 217-221, 2001.



. . Con on o n dg



. **2.** R o l o n o d on



. . Con on o ngl *z*

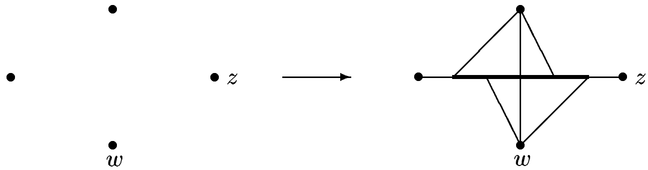
h or . k oto i 5) *g*
b sf m b s f s f g s
m s f s g m m m g

M r r, i is n n r c nn c d p n ri n i n c n
r ns rm d in n c dr n s q nc c n r c i n d s, pr s r in
c nn c dn ss [2 r d r s c n nd in [
p n ri n i n nd c
ri n r *g* is r m r d s , nd ,
id n i r r ic s , nd in sin r nd r p c r p irs
m ip d s r sin d s r sp c i , s s n in Fi n is
p p r, d rm ri n i ns c n r c i ns ri n s
in r r m in r s s

ro o itio . *g b sf m*
b s f f g m f x f g

h or 2. - *g b sf m*
b s f f g f
g x f g s g - ss

R m r in Pr p si i n nd r m 2, n c n r c in ri n s r
n n r ns rm ri n i ns in r dr n nd n c dr n,
pr s r in c nn c i i nsid r p n ri n i n in d r m n
p n ri n i n ddin r d r in c c E r ri n
in c n ins n d s, c n r c i n c n r c s , is
c n in d in s s p r in c c s in s, r c n r c i n
ri n i ds m ip d s nd c nsid r c nn c d p n ri n i n ′



. . bd d ng by o l ng

in d r m n p n *g* i , p n r p i c c q dri r p in p *t* n in c c nd ddin d s , *t* nd , r = , , , s s n in Fi n ′, r ri n s n d inc d d in s s p r in c c s s, c n r c in n ri n r ns rms ′ in r p i c is s s c r ns rm i n is r s ric d n r c r i n r ic s, d s nd ri n s , i pr s r s n n c m in ri s r c r s s m ric pr p r i s r r r i ns F r mp , sin r m 2 rs pr d r c nn c d p n ri n i n i ni c s r n n c dr n s s r i in m ddin in p n s c /2 c s r c ri n s [i s n s n in [r p n ri n i n i ni c s c n dr n in p n s / ir c s r c , nd pr r m 2 s s inc d d in [Pr p si i n nd r m 2 ms s s m in r s in nd s r n p c d s d i s r rm i i ns m ric pr p r i s in p n ri n i ns

f

c nn c d p n ri n i n nd nd n d nd c , r sp c i is n n ≤ ≤ sinc n ri n i n is c nn c d nd n p n r p s r d r m s s n d r sp , ri n is - *b* i r p , d n d / r sp / , in d r m c n r c in r sp is s i c nn c d p n ri n i n

L . *L b* - *g b* - *g f s* - *b f f f s m s* - *C s g m w V* = *f m w s bs s k m*

f s ffici nc is i s nd nc s n c ssi inc is n c n r c i , / s s p r in c c *C* r , r [*V* / risin r m c n r c in in i s n *C* s, c n p *C* = [*a* r s m r ic s *a*, *V* / [O i s , *C* c rr sp nds s m s p r in c c *C*′ in r *a*, *V* , , s pp s *C*′ = *a* , n n i r c mp n n s

C' is n r sinc V c rr sp nds r [V / n C s, mm s

Mimic in pr mm , c n s in mm s i m di c i n

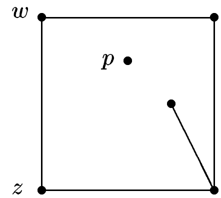
L 2. *L b g b g f s - b f f f s m s - C s g m w V = f m w s bs s k m*

pr Pr p si i n nd r m 2

f f s p n ri n i n r n r dr n s r d r , n r m , s pp s δ is c nn c d, n r ri n ndin c is c n r c i , mm 2 s, ss min s s p r in c c $C =$, s s r is s c n r c i ri n in in ri r C m s ss m C is *m m* , is, r is n r s p r in c c in n C C nsid r c in n C C, r V n C c in n C C s rin d = , n ss mp i n C, s d r , ic is c n r r minim m d r s, = r c im ri n is c n r c i n , n mm 2, s m d d c n in d in s p r in c c r, sinc ri n s nd s r s n r , s c c c is inc d d in n C C, ic is c n r r minim i C

f f m s ffic s pr i is c nn c d p n ri n i n r n n c dr n, n s c n r c i d incid n r d r , r c n r c i ri n inc is c nn c d, s n r d r ss n

pp s s r d r , , nd r n i rs in is c c ic rd r is n c n r c i i , / s c , n s c c r s m V , , , i ss n r i , m s pp s is in in ri r c c imi r , i is n c n r c i , n c n nd c c r s m V , , , rd n r r m, n c s d c r s in p n c nn cr ss n n p in , nd nc = n is c s , r p ind c d , , , , , = is is m rp ic n c dr n inc is c nn c d, m s n c dr n, ic c n r dic s ss mp i n s, s n d s r is c n r c i c nsid r c s n s n r d r mm , i is c nn c d, n r ri n is c n r c i , nd nc s pp s s s p r in c c $C =$ s C *m m* , is, r is n r s p r in c c in n C C ri n r c in n C C, r / V C sinc C is c c in n C C s rin i c / V C sinc C is minim r sinc is



. . o F

c nn c d Fi c im ri n is c n r c i d s , n c c , mm , c , , is n c n in d in s p r in c c

F r in c n r dic i n, rs s pp s d i s n s p r in c c C' m s pp s C' is n c n in d in n C C F r r is , C d n minim , c n r dic i n O s r n i r n r r d c n i F r r is , i , i is d c n i , n d s p r in c c , c n r r C in minim s, $C' =$ r, i s , n d rm c c is imp i s is s p r in c c r d r is c , c n r r in c nn c d r δ $>$

F r d s nd , n c nsid r c s n i s n s m s p r in c c $C'' =$ a r s m a, V , m ss m C'' is n c n in d in , minim i C inc is n d c n s m r m n s , r a n C'' is in n C inc C'' is s p r in , a / C'' s a / C'' sinc C'' is minim r m inin p ssi i i is a C'' nd $C'' =$ a inc q dri r a c nn s p r in c c minim i C, i s di n i r r a , n d c , , , nd i a , n , , d c , r d r d c n c s s, c n r dic i n s, c is c n r c i

efe e ce

1. . n o, . nd . W n b , On n b o ngl s n s g l n b dd ng o x l l n g , *m T S B* 1999 , 110 11 .
2. . b y s nd . o o, o n on.
. . b y s , . o o, . Od nd . W n b , ngl s n onn d x l l n g s, n .
. J. y, n x l d d no o o o d on, *T* 1999 , 9 100.
. . o o nd . g , n ng ng l ons on los d s n d g l s , o n *sc t t* .
. . n z nd . R d , *l s* ¨ *T* *l* , ng , l n, 19 .

S S s S S s l s

D v d p c d Be pec

p n C p n
n s y s C l b

a . x n n s lls n n b n n s pl p ly ns n n] n p l p s n n n x ns T s p s s fin n n n n p l pp x n n x p ly ns T s p s n n n b xpl s p n s ns pl x y b n s n lls n n p bl s s n l] l p s n x p ly ns p n ly s b s s n l p s n n l p n p n n p p ppl s ll ns n l n n ss ly n x) p ly n l bj s

t o tio

Coll o de ec o b c d vo d ble co p o l p oble ll e o eo e c odel volv objec o o ve we e c e o o objec e pl e e co ple o coll o de ec o o be co ple el de ood

Co de o e ple e p oble o de ec coll o be wee ov pol o l objec e pl e Mo pp o c e o coll o de ec o wo wo p e F b o d p e l e l o d d c e e ed o de e e p o objec po bl coll de d e e ow p e e e e l o d d c e e e e c p I e e l c pp o c e o ce e objec o ep e e o be ed o o e o e wo p e o volve o e d o b d ep e e o o de ll ed o e e

ece l be o e c d c e D ve bee p o po ed o wo d e o l coll o de ec o e e e p o vo d b d c o be wee b o d d ow p e d e d d p o c e e e d de ee o ep o be wee objec e c d c e e plo e co e e ce o e l o o d ove e collec o o el e e eo e c e *r i* o e e ce e objec e d jo Objec e ed o ve ed b c e ble o o jec o e *igh* d ce c e e ed p o q e e b ed o e e o e p o c lc l ed o e c e pl e ce c e e p e – e o lo e old – o w e pl c ed e d c e be pd ed

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 222-236, 2001.

ee l p e e e ve o de c be e co ple o ve co
o o objec k – e be o objec – e o l be o ve ce
d – e ze o e i i i k di i i ep e objec

e c d c e d e oc ed e ce l o c be
ev l ed d co p ed w e pec o o de ed c c e c ood
D e l le p ce dd o o e p r i e
d c e v c be e o ed q c l e e l e o ce c e
c be pd ed e l e pl o objec c e d
i e wo c e be o eve dled b e d c e o
ve o o ll co p ed o o e wo c e be o e e l eve
be dled o o o

O p c l elev ce o e wo o p pe e e p pe o B c
e l w e e p o o ece l co ve pol o e ed
b l ced eode c l o d e e o o collec o o pol o b
w l e l e be o ce c e ed e e c e e
lo F e o e e ve e dd o l e e e e c o
c l e c e depe d o l o e c e e o e pol o d o o
e o o o w c c l e o v l l o e e c e c

I co e e c ep o c e od ced
ce c e e o ze b o c o c l e o depe de
d e ce d c l o l e e o e c e c

U o el bo c e de c bed e ble o e plo e d gr o
ep o o e bo objec w c c c o e po e l
e c e c e c ow p o e ple e c e ce c e e
ce d jo e o wo w del ep ed l co ve objec w
o l o ve ce eed o be pd ed e de p e e c
co be o ce c e wo ld ce o e e e o o

O e pp o c o co po e c o o o e c ep o
c e volve e e ce o o o o d Fo e ple b
e l d e e ce o e co p c o o o d o Mc ll e
e l o e o d jo co ve pol o ov e pl e le
c e p ov de cc c p oo o d jo e doe o e e pe e
o po e ll l e be o ce c e pd e eve w e e objec e
w del ep ed F e o e o obv o e e l z o o collec o o
o co ve objec

o e l pp o c o e plo e de ee o ep o o objec
o pp o e e c d v d l objec b o e d o co e o e pp o
o w o e co e e depe d o e ep o e d jo e o e
pp o o pl e e d jo e o e objec b e o e be
le e pe ve o Fo ol ed co ve objec v ewed e e ec
o o l p ce de e ed b e bo d ed e po ble o co c
e c o cce vel co e pp o o b ed o e ple de o
p o e ve el o o l p ce co l e ple o co
p o l eo e e Dob p c pol ed l pp o o e
c c o e l developed e c coll o de ec o c e

o p o objec b ed o d ce e ve v o c e c e w o e pe o ce e eb de e ve o e de ee o ep o be wee e objec el ve o e d e e

Fo co ve objec e co e o o e co ve objec o o e e e ll o collec o o d jo b po bl e le ved o co ve objec o ed el cle w co e e l o o o e c c l ep e e o o ow c o o co ld be e plo ed o p ov de ep o e ve co ple bo d o e c coll o de ec o

O objec ve d co p o p pe 5 o de o e e c e le d el l w o od c o p ov de e e o e v o e de ee well e e o e ep o o collec o o ov pol o I p pe we develop e o d o o c c e b oc o collec o o co ve pol o e e plo e c c l pp o o b epl c co ve c w p e do le b e c c l pp o o c e co eq e ce we c eve ed c o e be o ce c e pd e eve well o e e c e ple e o o e p e do l o pd e p ve o e co e o e

e o e e collec o o c pp o o p ov de pp o o c e o collec o o ov co ve pol o d p ve e e e d v d l pol o e ep e e ed level o de l c e o ep e e o e d c ll c e o e bo e o l ep e e o ze ove e e e collec o o pol o p opo o l o e be o objec depe de o e co ple o e d v d l objec

e objec ove e c e level o de l o o p ov de clo e pp o o e w e e e pol o e clo e o e c o e w le co e e w e e e pol o e well ep ed e pp o o ed c w co o o e p o o e ee p ce o p e do le o c ll p ov de w e e o de ec e p o e c e bo d o e pol o w e e we c e e d e pp o o o w e e we ve o c e e level o de l

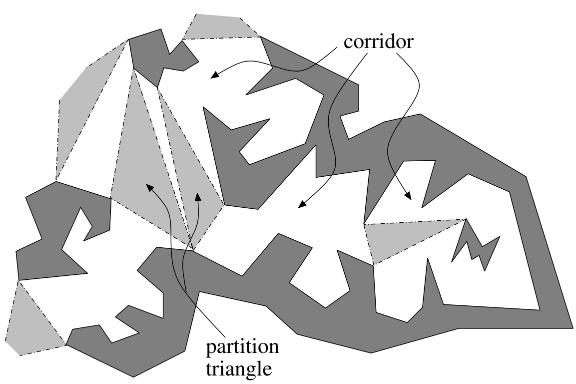
co ple e l o ee c e c o o e c c l l e o e od ed c e e c lle e eve ele po ble o p ov de o e p l e l de o e e po e l o c e I p c l pe o ce co p ble w o e ep o e ve c e o c o e l w de ed o e ep o o co ve objec pec c ll o co be o d jo co ve pol o e pl e le be e d e e le be e o l be o ve ce d le be e d ce be wee ll p o pol o d e o o O c e p oce e lo lo eve w e e c o e pol o ove l e jec o ll eve c be p oce ed co o lo lo e pe eve

I e e ec o we ec ll e po e e o e e c ep o c e ec o e o e e e e l p ope e o e co ve c pp o o e c e eeded o ep o e ve e o o e d ll e eve l po e l ep e e o ec o dd e e

e e o o e ed w e p o e d c
e ce o e c pp o o cco p c e e p e do
l o o e ee p ce d e ew e c eve cco p e
od c o o pp o e pol o ec o 5 ze e v o e c
b e o o ep o e ve c e F ll ec o e o o e
d ec o o e wo e P D d e o o e eco d o c e l
p ep o w ll co e de l o e l p pe d el ed
wo

e i eti e tio t t e

I ec o we ev e e c ep o c e w p e
e ed e p z o e e e p c d p o ep
o e ve c e we de c be ow e b l
o b l o o e ee p ce ep ve e o
k d jo ple pol o e pl e o e e w po
e ve o od ce e e e d c pl ed o I l o e cle
e e e o w c e c e c o c l d e w c e e e
ece o l o e e c e c oppo ed o e co ec e o e c e
c be co c ed b de *k* *r i i* *ri g*
P o le e ed o e c e p o e p ce be wee
objec e co ple e o e pol o d e po o d jo
e o c lled *rrid r* ee F



i . . p s n sp b n p ly ns

. *r* *rrid r i* *d d* *dg* *r i i* *ri g*
e d ed e *d* *i* *d g* *r* *h i* *dg* *g*

e co d o e p o le po ve e
c lled *ri g* *r i* l l *rrid r* *r i* e o e
o e pol o l c o e d ed e o co do ove l p I ow

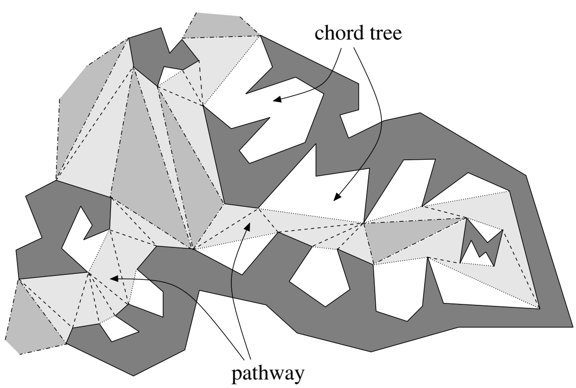
e e ce c e ce o de ec ll c e e opolo c l c e o e co o o pol o

. *h gi r r gi g r i g i i ri d r i h g h d i ri g r r rid r r i*

o e bl d co do ce c e d c e b l de e c co do bd v de e e p p ce be wee wo de pol o l c d co ec wo e d ed e o eq e ce o *d ri g* ee co ve ve ce *r r* jo ed b ee co c ve c

Co de l ed co do w wo bo d pol o l c l beled d δ wo d e e d o l o ed e c be d ed *ridg* co ec w δ d *h rd* co ec w o δ w δ

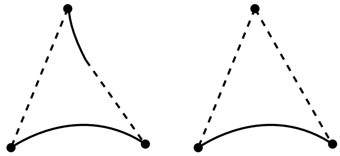
e le dj ce o b d e d e d ed e o *h* ll ed F I e p w e oved o co do w e po bl e p e o e p poc e e c bo ded b c o d c lled e *r h rd* o poc e I e d l p e ed e d l o c o d d ce ee c lled e *h rd r* e c poc e ee F



i . . p s n s n p ys n s

p w d o be *d g r* o e d ed e o b d e ove l p w c o d o pol o ed e o co do ce c e ce o e co e po d o de e e e p w I l w e o de e e c o p w ce ed b e o de e e c o e c o co e p e do le

I o de o llow o e c e pl pd e e e o ce c e w c ve e ce e d jo e o objec e *i* o ld be ep ll po ble o ed ce e ze o e c ve e oc ed w e c co do e le oc ed p w e e ed o e o p e do le e ze o e e l e l e l el ed o e ze o l ep o o e wo pol o l c de e co do



i .3. T yp s bj n l s b s/ n s s)

p e do l o co c ed d ed b d
wo pe o p e do le ve p w

– *r r ri g* ve e d ed e b d e o o l o e de c o d pol o
ed e o e e objec o e o e wo d
– *ri g* co e d ed e b d e o wo de d c o d pol o
ed e o e e objec o e d

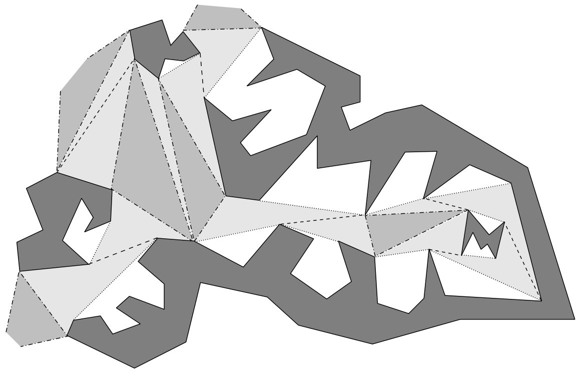
o e eve objec le p w le o e de co
o l o b d e o e d ed e ee F
e p e do le ll le p w e e ed o w pped
o de o e bl e ollow v

n nt (t n nt . *hr i d ri g i h i d r r ri g d d ri g g di r*

ee F o ll e ed ve o o e p w o o e ple
co o o e p w v
e o de e e c o p e do le c be ce ed b e o de e e c
o le o ed e c o ee co e e p w v
ee e c ve e o ve co do p o ll dd ve co
ze p opo o l o e be o co e e oc ed p w
el ed o e ze e e l ep o o e co do
o e ollow

. *i i i k r r r h di g h i rrid r i , ddi i , d i i g h r r r r ri g i id h i d h*

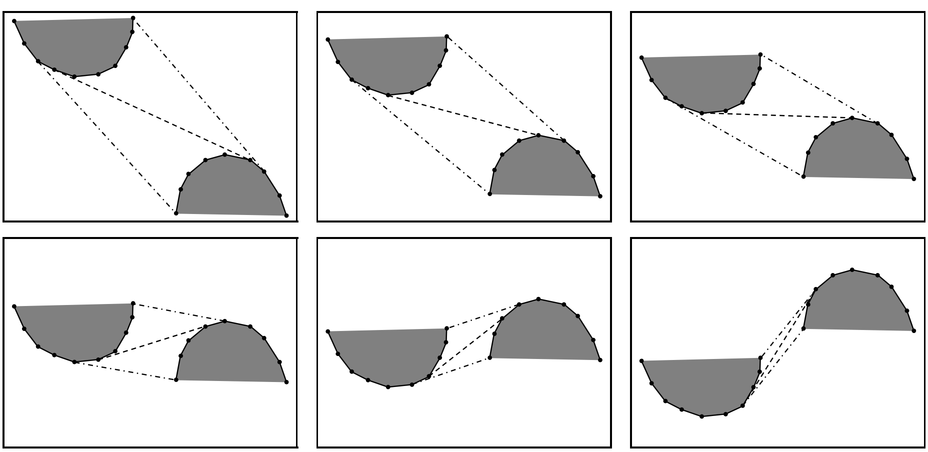
l o e be o ce c e e c ve e bo ded b o e
clo e o w co ld be ed e e l o e ce c o o d jo e
o d o ee e be o ce c e pd e eq ed o eve ve
ple o o c v l e ceed w wo ld e o bl be de c bed ece
Co de e c e o wo co ve objec w o l o ve ce l
lo l e ee F 5 e objec ove e c c o d d e d ed e
o c o e c ve e o e c o e pol o co e po d o
ce c e l e I ee obv o co be o ce c e p
d e o ld ce o ce d jo e e ce e e o
e ep o be wee e objec e e l ep o doe o



i . . n l ps n l n p ys

c e d e o o d e ep o ll e l e w e pec o e ze

I e eve o co ce p pe ll objec e co ve poc e d co e le do o e I ollow o I v eve p w co o o wo p e do le e de o w c co co o e o e bo d o o wo objec I e e ec o we d c e co c o o d c e c c l ep e e o o e e e w c e po ble e o l o ep o e ve bo d o bo e e po ve e d e c e c o e c e



i . . T n x bj s n l n s l n s

ie i e e et tio o o e i e ti e o e

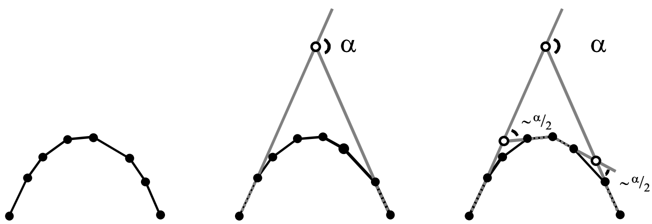
O e o e d e l e o o oc e o o co ve objec e e d ec l o o deco po o eo e c odel e c e le d e elve o e c c l ep e e o e c c l ep e e o o c co ve objec pe e c e e po e o b c e ec o w e pec o po o l e d ep o q e e 5 9 7 d coll o de ec o q e e o ele e o o I l o po ble o b ld e c c l ep e e o o o co ve objec e o e co ve p ece o p ov de e po e o ep o q e e w o e co c o o e co ple o o e ep o e ed e o e ze o e l ep c 7 I e co e o e c coll o de ec o c o e l de c bed e e o e c c l ep e e o o d v d l co ve objec o p ov de ep o e ve e ce o d jo e ce c e o p o c ob jec de v e o o o odel C c l o ppl c o dd o l co o o e e c c l ep e e o e be d ce e ve e e e e d ce o cce ve level o e pp o o e c o e de l pol o ow co olled o bo ded b o e e po e l c o o e level de e e o e ple wo co ve objec o d e e o d w ep o d d jo pp o o w e e pec ve e c e o ll o lo ed e

e e c c l ep e e o o c lled boo e e c e p opo ed b c o e l depe d o bo e co ve d co e depe de ce o e d v d l objec O o l o b e be e o e c c l ep e e o o c e co e w c we ve collec o o co ve objec o o I e e e c c l ep e e o o d v d l objec be co e depe de d lle ble

e c eve co e depe de ce b b ld o e c c l ep e e o co j c o w e e c ep o c e de c bed e p e ced ec o e e l c e o e pp o o o e c objec e collec o F e o e e e o l ze o e co b ed pp o o o w ll co l ple o e o l be objec depe de o e co ple

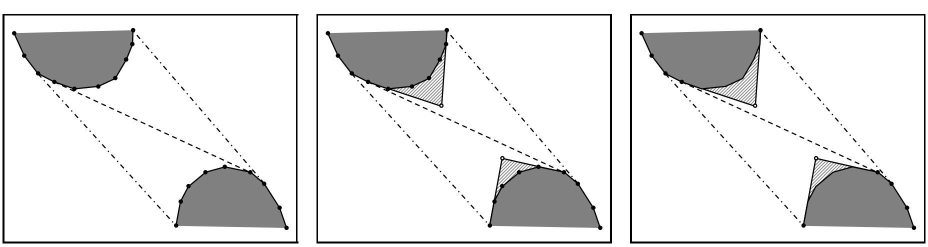
w e c e w c o e l we c e e o e c c l c e o co ve pol o l c e ce b c p ope e F we eed d ce e v de ed bove I w ll ce o o p e e p po e o ve e c c l ep e e o o c e l zed e q e ce o lo l e w o l o ed e w e e e ze o e de l c e *h i* e l e e c el d e + l e co o o e ed c o o e be o e e o p edece o F e o e eve e e o e l e co po o d ce o ^{i} o e b e c w e e de o e e d e e o e b e c e eco d p ope we eed lle b l o pp o o o o e pol o de p o po bl p ece

w c eed o be pl d ppe ded e objec ove el o o
e bo Fo o l we eed c e pe e c c l
ep e e o o c o be deco po ed o op level bc
e F e o e ppo e c e e o e ep e e o
o dj ce co co ve c o o ep e e o o e c o ed b
ppe d e wo e ed e dpo



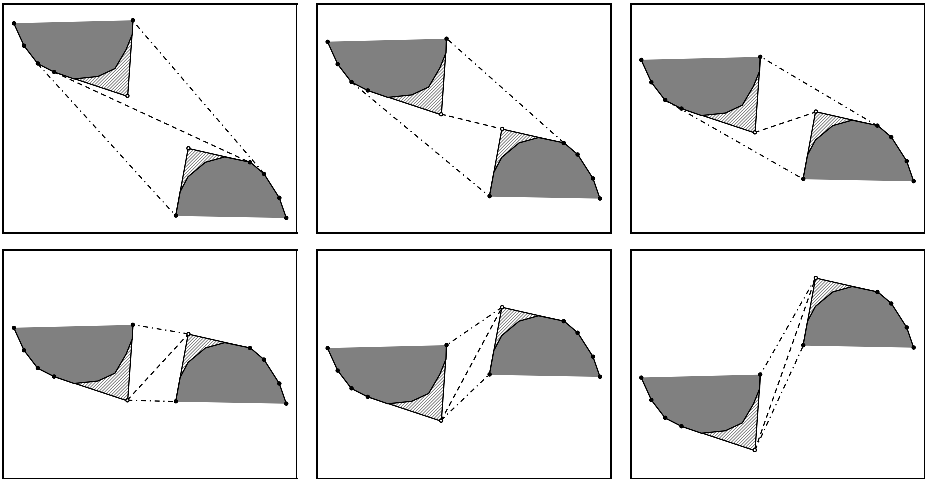
i . . C ns n y

o e boo e e c e p opo ed b c o e l co ld be ed
o be e l d p ed o o p po e eve ple e c co c ed
b oc w e c ve e e b e c e cep e l e e e l
le o ed b e e e o o co ed e d o o ed e ee
F e ce w le α e e e epl c ed $\lceil \alpha \rceil$ e
b e e o o ze o le ed e pp op e le o eve b e
c ve e le o
ce e objec be ep e e ed e ll co ve e el ve co ve ll o
d v d l objec eve c e pe pl c e c c l ep e e
o w e e e e e bo d o e c pol o e ble epl c o o
ve ce bove ep e e ed ple I c e e c b e c
ep e e ed b co be o po e w pd e cl d e e
e d e c c l ep e e o ple e ed pl c l
b e c



i . . C n pp x ns n x ns bj s n F 5

e l ze o ep o e ve c e b epl c co ve
c e c p e do le b e c c l ep e e o ee F 7
oweve e e o o e e c c l ep e e o p e do le
bo d e o ed do o wo ld eq e v eve p e do
le co e be e l oppo ed o pp o e pol o ve e w c



i . . T lly p s n bj s n l n s l n s

wo ld e e c o e po e l be e o e e c c l ep e e o
I e d p e do le e ee o c b pp o o ve ce
bjec o l o e co e be o pp o o c co ed
w le p e do le o o e ed co pp o
o c e o e e pl ed be wee dj ce p e do le d
e ed ll e b ble dj e o e po e o e
c e ee F 9 o de led e ple

F epe e o o o e objec o F 5 o wo e c c ll ep e
e ed objec o e c e o l co be o p e do le
c e occ

M i te e

e ow e we e ve k co ve pol o ov w co o
o o d we e ble o co p e e l e e o ce c e o
c ed w o e pol o co e

ce we do o d be wee pol o d pp o o ve ce
e c e po o e d ed e d b d e ll o e pd e ope o
de c bed c be e ec ed be o e e ce c e l we e
ble o pd e e d c e d e o e e v b co p o o
co be o p e do le o e pp o o c e
eloc o o b d e d e d ed e d e e o o o v c be
pe o ed co e o ed e l e e e c o e e c e
o e p e do le w e eve co ple e b e c ove o o e le
o e e c be do e e b e plo e pl c e o o
e c c l ep e e o

e e c oweve od ce wo ew eve pol o ed e o c o d coll de w pp o o ve e d pol o ve e coll de w ed e e d o pp o o ve e

ve o pe c be dled e b popp e pp o o ve e d e o e v e co ve c w p ev o l dde ee F 9 o ll o d o e ow e e c e e coll o

o e be o p o le c be c ed o pp o o ve e w e eve popped ce o p o le o be c ed o pp o o ve e e eloc o o e p o le o pol o ve e co b e o l o zed co o e c eve ve o pe eq e l ple pp o o ve ce o w e e e c o be popped o de o de e e e l coll o oc c ed ow l e o e e c ve o be ve ed d ec l p opo o l o e ep o o e objec w e pec o e d e e pec c ll e d e e o e objec d e c l objec ep o e o e e pp o o coll de e lo lo l e o e e c w ll be ve ed e c e e c e w ll e e

5 i eti D t t t e P o e tie

I ec o we l e e p ope e o o e c c l c e d l e d e e ce w e o l c e ec ll we ve k co ve pol o l objec co o o l o ve ce o e e l bd v o o c collec o o objec ze k O d c e e ollow de ble p ope e o D

p tn

e e e k p o le e c o w c w ll co b e ce c e o e c ve e c pol o co b e ce c e w e e e ze o l c cle ep pol o o e o e objec e e o e e c ve e co k ce c e e

t

Ob e ve e c pol o o ded b c cle o p w co ec ed b p o le D e o e we c e e o e e e be o ce c e e c pol o ppe k e wo c e

p n n

O d c e e wo d o ce c e le ce c e d co do e co e ce c e o e e c c l ep e e o e pd e o bo d o ce c e c be do e lo e w e e e be o ve ce o e o ve p e do le volved e e c c l ep e e o e co e po d eloc o o b d e d e d ed e well e e o o o v d pl c e c o e e c c be cco pl ed e

I ll c e e pd e volve e de c o d c e o o co be o ed e d oc ed co e ce c e c co e d le ce c e d bed d p oce eed o be de c ed led o e c ed led e eve q e e w c e dd o l e lo c e ze o e c ve e

ffi n

e pe o ce e e o e c d c e e be o eve p oce ed e wo c e l e d po ed o b w l e l e o o o e c e c e c e o coll o de ec d c e o cle ce e e o c o c l d c e e b e w c o co p e e pe o ce o e D e o c o c ll de ed w co e e e l eve

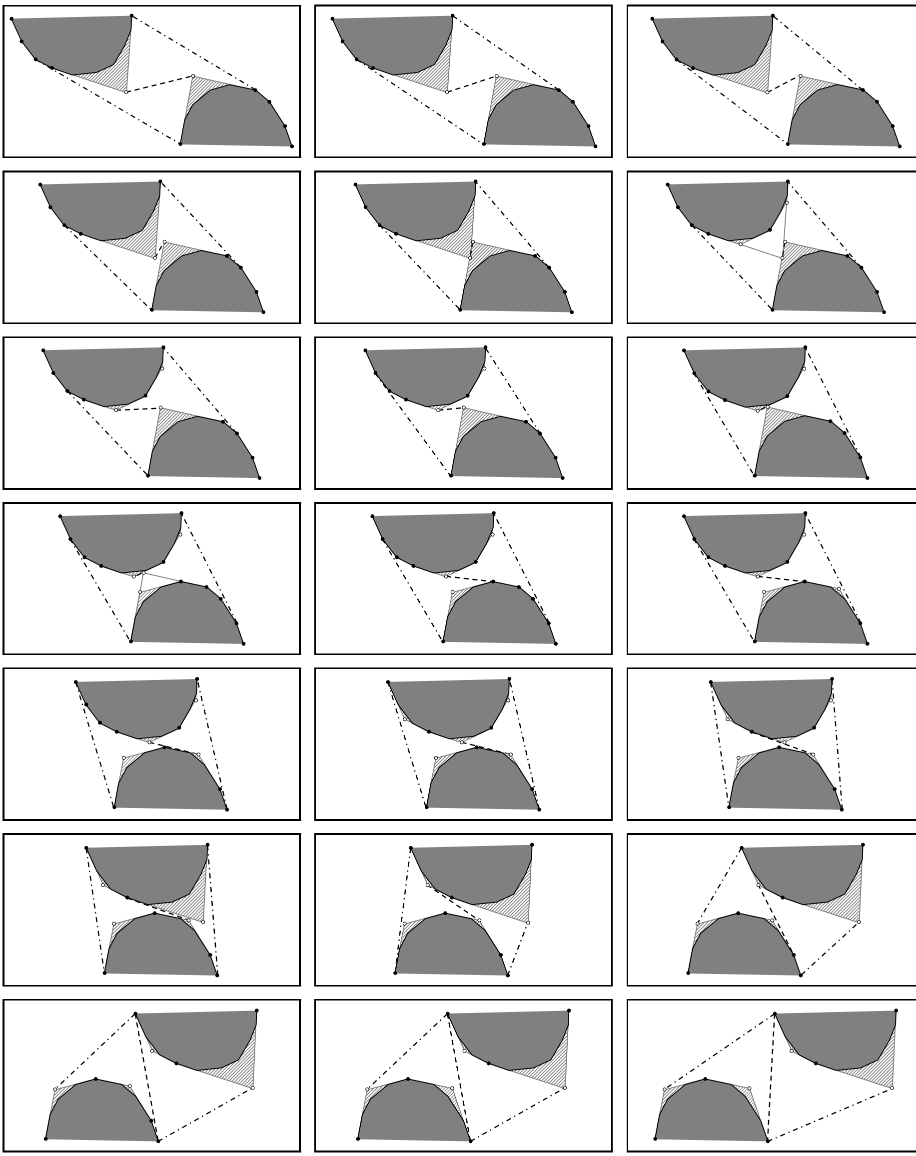
eve ele w e c e w o o l c e po ble o ve e ppe bo d o e be o ce c e l e e pol o l e lo l eb c jec o e o bo ded de ee

. *k i g i h r i r g g r i r ri d gr d h h r r d h i d* [3]

B od c e c c l ep e e o o co ve c e e c e c o o c e beco e ep o e ve I e c e o co be o d jo pol o e pl e e e c e c co p ble o e bo d ob ed b c o e l e eve o e e c ed c e w e e be o co ve pol o e c ed o wo e be e d e e o ll e pol o le be e o l be o ve ce d le be e d ce be wee ll p o pol o d e o o

. *h r r d i* lo lo *h h i d g i i r r r r d* lo lo *i r*

r e c P o le d b d e w ll o l c o ce o eve ve e o ve p w e e o e c e o bo d e be o ve ce ppe o e p w d e o e e c d e o o e e l ollow o e c e e e o ve ce e e c d ce le o ve pol o ⊔



i . . Flyp s bj s ll s n p l p s n n

t e o

I co p o p pe 5 we e plo e e e e l z o o e e l o p pe o collec o o o co ve objec I dd o we ope o ob e e c e c bo d o v o d o o o b e plo e p ope e o o e c c l c e e pl c o o c e ee p o o e e o o wo d l o ee d e o b e l w ll do b edl be eve o e o c lle e

Re e e e

l s b s n s b
b n s n pl n s b s ns n *P* *M S* *s* *G*
p s – 5
l s b s s b n n bl
sp l n s n ll s n n n *P* *W ks* *g*
d *s* 000
s s n b s s b n n n ll s n
n s pl p ly ns n *P* *M S* *M S* *s* *s*
g *s* p s 0 –
s b s n s b s s b l n *P*
M S *M S* *s* *s* *g* *s* p s – 56
5 C ll n b n n s n n x bj s n n
ns ns *M*): – n
6 T C n C s s n n s *d* *g* *s* T
ss C b 0
b n s b p n pl ly s n n
l ns n p n p ll s n n *P* *s* *S G*
S *s* *g* *s* l 50 *s* *S* p s 65– 0
p n l 0
b n n p F s n p ly l n s n
S): – 5
b n n p l n l n n
s p n n x p ly *g* *s* 6: – 5
0 b n n p n n s p n p p ss
p ly — n fi pp n s n *g* *g* *s*
d P *g* *g* l *s*
S p s 00– n s y n l n 6– 0 ly 0
p n l
s n b s lfi n n p n s ns ll s n
n n x bj s n *P* *M S* *M S* *s* *s* *g* *s*
p s – 6
b s n y n n n C p n s n
n x p ly ns n *P* *S* *d* *v* *W ks* *g*
000
b s n s s: s p n *P* *d*
W ks *g* *d* *s* p s – 0

p n y n n p nn n ll s n n
s pl p ly ns n P M S s G p s – 0
000
5 p n p nn p n s ns n ll s n n
s pl p ly ns n p p n
6 ll s p n n y n p p s l n
n n x s s n pl n s G 5: – 05
6
n n s n n n s p s s pl p ly ns n P
M S s G p s 0 –
M k s g s d d s s s p
C p ns p ns n
T T ss n s p s l s nsl n s p b l y p ly ns T
n l p C 5 l C p ll n 5

c T g s d s

ir s i e r

k rs s ara Ok awa a a

bs ac ac r a la a l s a r a la ac r a l s) r r a l s c ssar r a ac r a la a l n) a al) r all n s W r 4) = 1

O O

r *ul* p g n e e n s i isi n e p g n in fini e n er n n er pping ri ng es in s n is in ri ng es re ei er is in e sing e er e in n r e ne en ire e ge in n er r sse n ffer pr e e er p g n i s ri ng i n in *us* ri ng es n ern i e n pper g e n g ri r ri ng ing g ns in n n se ri ng es An *u* ri ng i n p g n is ri ng i n se ri ng es re e ri ng es en e s n e ri ng es re ne ess r r n e ri ng i n n se ri ng e is pr e s pr p se r in r ner in 6 see pp – n e n ei er 6 g e s i n e n er is C ssi n r s e s re n e ri ng e in eig e ri ng es eig is e ini n er e ri ng es r s re n e ri ng i n in eig e ri ng es is ni e in sense D es e er p g n i n e ri ng i n is is ns ere r i e in e r

r p g n e en e e ini n er e ri ng es ne ess r r n e ri ng i n n e en e e i e r g ns s 6 n is p per e pr e

A er e p g n is e

n *u* *r* *r* i _

n *us* *r* *r* i _ n

r *r* i _

ere _ en es e in eri r ng e e p g n e e n e ri n g i n p g n en er ri ng es in is e e *s* s e reg r s p ne gr p is p n r gr p e e e in e p ne A er e is e

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 237-243, 2001.

r r r i is er e
s d r i ies i in si e n
n *r r r* i ies insi e
n gr p e n er se er i es e egree is en e e
ing e i e e r

L u r ul h

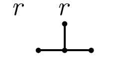
h u d r s l s

eg $\begin{cases} s \quad r \; r \; r \\ s \quad r \; r \; r \; s\,d \; r \\ s \quad us \; r \; r \end{cases}$

$\leq$ *r rs d* $\leq$ *u r rs*

L u r ul p ly d supp s h
h s s l r r r d $\leq$ $\leq$ *h s pl*

r ph s rph



r e e in eri r er e n e eg e e e s gr p
in e n i s neig rs en is ee n is p ne gr p
ine r ing s e ri ng es r e si e e in
i e in $\leq$ $\leq$ e ri ng e
r e si e n e ge en e n er e re ses ne
e re se r er e e ri ng e n e ge se
en s e egree s ge p ne gr p i n ne in eri r er e
n s is ing $\leq$ $\leq$ e egree s e n s
ri ng es s e e n n en e ges is pr es e e

r n ε e ε neig r p in is se p in s i in
is n e ε r e ing e i e e r

I C s u r l h h r s ε *su h h r*
y p h ε *h rh d h r l C s u r l*

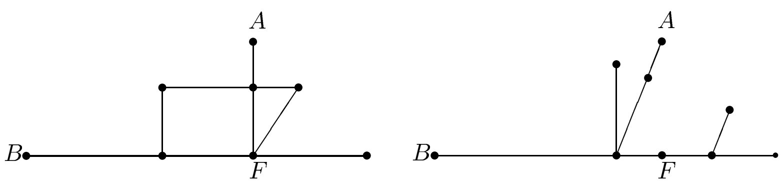
e re e is ne e s i n n ei er 6
n e re ers e re ining is pr g e re is n
si p er n is rg en i n e pp ie e ri er se

r *r y u r l*

r e e e ini e r n n e ri ng es n e C e n n e ri ng e s C e e n e ri ng i n C si e en s e s ne in eri r er e r er ise n e ge e n ing r e n n e rner C i i es i in ri ng es e s ne i is n n e ri ng e in e r n n n e ri ng e e e C n r i i n
Ar n n in eri r er e ere re e s ri ng es e en e i s r re in eri r er i es en e si e is e s n ins n ne in eri r er e en e si e is e

r *L* C *r* *l w h* *u* *r r* *h* *r y p* *h s d* C *h r s* *u r* *ul* C *w h s su h h* *s h* *ly s d* *r* *ly* C *d* eg C

r e e e perpen i r r e si e C
irs s pp se $\_$ C is e e R e e p in n e si e s R is p r e C e e in erse i n n R S e e perpen i r r R e si e C see ig re en e ine seg en s S R n RS i i e e ri ng e C in *us* ri ng es s i e S r s ig n s i e r C s ig s e ri ng es R S RS S C e e e ri ng es en e e n s i e s ig in e ire i n s ri ng es e e e ri ng es



a))

g 1

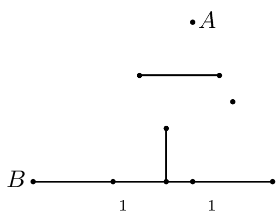
e s pp se $\_$ C is n e see ig re r ri e p in R n e si e e S S R e e e perpen i r r R e si e C R e e in erse i n S n R R g es ne r en $\_$ e es se ere s i R g es ne r en $\_$ e es e en e ere is p si i n R e een n r i $\_$ i R n S S R R s p si i ns n e e p in n C s S n re p r e en sin e $\_$ C $\_$ is e ies e een n C en e $\_$ S is e s e ine seg en s S R n RS i i e C in se en *us* ri ng es e e e i p in sin e S is n e ri ng e e n s i e s ig in e ire i n perpen i r n r s S re ins e n e ri ng e en R e e e ri ng es n $\_$ R $\_R$ S e es ess n in s i e R r s ng es $\_$ R n $\_S$ R e e e en e e esire e ri ng i n

r ry 6) *r y u r l C C h lds d*

e i nee s e i i n e s ri ng es r e ing i ri
er s in e ne se i n

L C u r l h r y p h s d
C h r r w p s R d R C su h h h l
s s R R d d C u r l s

r e e e i p in *C* en e n e s *R* e
i p in s n *C* respe i e e ss e ies e een
n e e e p in n s is p r e *C* see ig re
e e e p in s n *C* s ‖ n ‖ *C* en
sin e ies e een n e ine seg en s n in erse e
e e in erse i n p in n e *R* e e perpen i r r
C en i i e e s see e ri ng es *R CR* *R* re
e ri ng es



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r ry *L C r l d l p h s d C I*
– *s u h h r s u r ul C w h s s*
su h h s h ly s d r ly C

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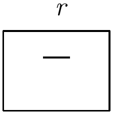
r *r y u dr l r l C C* ≤ *h lds*

r r n re ng e ere is n e ri ng i n si e s s



en e e ss e *C* is n re ng e en i s
e s ne e rner s *C* is n e ri er e ss e
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C i i es *C* in ri ng es

irs s pp se –

e e e perpen i r r *C* in e – is e C r
r i p ies ere is n e ri ng i n *C* i si e s

s is e n er e ing n C s C n e ri ng
e in s n n se ri ng es ng i e ri ng es er n
C re e ri ng es s i e e p in s ig in e ire
i n e e en is s ien s en e e ge n
e ri ng i n C in s ri ng es i e in is e
ri ng i n n er e ppe rs n e si e C n n e si e
pp se _ is e e ss e C is n s r er n
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ere is n e ri ng i n C se si e is s n n
er e ppe rs n C in e C is n e ri ng e e e en n e
ri ng i n C in s ri ng es

r *h r s* *u dr l r l* C *su h h* C

r e C e n e ri er s C
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$$\_ \quad n \quad \_C \quad n \quad 6$$

en C is e n e rner e re g ing s C
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ri ng e C n s n e ge r C in e ri ng e C s i s
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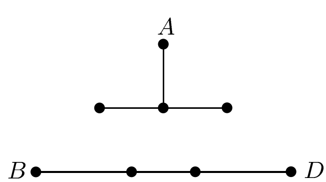

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in e ss e C re ssigne s in ig re e e
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e e e in erse i n e ine n e si e C e e in erse i n
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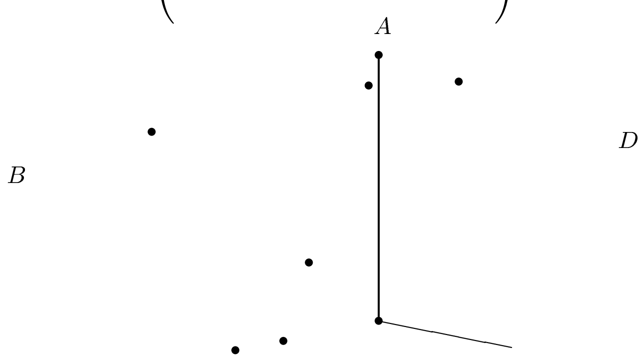

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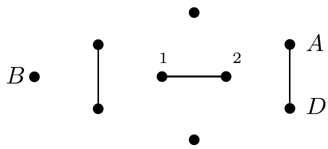
in e $\_S$ $\_$ is e s ie in e r pe i C in e
$\_$ C $\_S$ C n $\_S$ C is e s ie si e e ir e i i e er
C s s ie ei er in e ri ng e r in $\_$ e ri ng e C
sin e $\_$ CR $\_$ C n sin e C C nn ie
in e ri ng e en e ies in e ri ng e C in e s ie in e
s e si e s R i respe e ine C r er ise ne $\_R$ C $\_R$
is n e s ie e een n i i r sin e R s ie in e
s e si e s C i respe e ine S R s ie e een W n C en

sin e W λ W 6λ r s e λ _ W is se n en e _ R nn e e n r i i n

pp se s in eri r er i es en e re en r er ise e si e is in e $\leq$ $\leq$ in i is n e i see is is rp i e gr p i s r e in ig re in e n



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bs ac c r c r l c r ra cs a ra c r a s s s s c arra a s *n* s s a ca c l r r a l s a r ar r r r l l cr ss s W a sw al r a s s ar c r c s a r s al *n* l *n*) a *n*) s ac a arra w r c s r *n* l *n*) r O r al l a a a ca a

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O O

p r n ses e pr e s p g n ipping in p er gr p i s e n per i ns in D p er i e esign CAD CA n p er in ge gr p i in r i n s s e s n e s r e s e pr e i ing n rr nge en re n e ine seg en s in i ere re n re re r e e r ssings

is pr e is en e firs fin ing in erse i ns ine seg en s en i ing e rr nge en i re ire s r ing in erse i ns ng e ine i s n ne in e firs s ep e er g ri s e een e e pe ppr r ie e e p i p sensi i e r nning i e g in e gener n re se r fin ing in erse i ns n r p ing rr nge en s 6 n in e re e se r fin ing in erse i ns

is s rprising i g r n ee rre i p e en i ns ese g ri s ne re s n is rge n er egener e ses in i n en p in ne ine seg en ies insi e n er r e in erse i n ine seg en s is seg en r er n p in A se n re s n is ssi seg en in erse i n g ri s se pri i i es i re i e ig ge r i egree 6 n gener i ine seg en s re spe ifie e r in es eir en p in s en p ing n rr nge en re ires r i es e inp pre isi n n p ing r pe i i n e rr nge en s in e en e n s eep re ires fi e i es e inp pre isi n ing p in i p e en i ns ese g ri s i en n er r n ff err r si n res ing in in r e p ris ns n in rre res s

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 244-251, 2001.

n e e ping ne g ri ere re r pri r i s ere ini i e
ri e i pre isi n n e eff r n e egener ies e e
pre isi n is e er i i fin ing in erse i ns n e ns sin e es ing p ir
seg en s re ires e e i n n irre i e r i p n i 6
n e re e se C n’s r pe i s eep n p e in erse i ns i
n ree i es inp pre isi n iss nn n n e in s e ne
re e e pre isi n re ire en s C n’s g ri n is p per e es ri e
re s e ri ri n g ri pr es e rr ng en n
s e in erse i ns n es egener ies re ing seg en s e
s seg en in erse i n g ri s e er r e gener se r e
re e se i r n p sensi i e r nning i e g
p e in erse i ns seg en s e ere i r seg en ree e ni es
n n in erse i n in g i e n p e n rr nge
en irs n n fi n n p e _ n e represen i n n
rr ng en r n in s p i g i e r g ri n
n in erse i ns n p e n e r e rr nge en in p i
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s

r g ri ses p ne s eep i rns e s i D pr e ine
seg en in erse i n in n i D pr e in ining e is seg
en s in erse er i s eep ine s e s eep ine es r e rig
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g ri en efine i ness p in s i p n e r e in e in ri n s
r r g ri

nt y tt nn p

en e nn’s ssi s eep g r in ins in ri n s s i
es s eep ine r e rig r ss e p ne: e r in e r er
e seg en s in erse ing e s eep ine is n n n in erse i ns
e e e s eep ine e een rep r e n erse i ns e een seg en s
re e e se es s nges in r er s ese in ri n s s e p e en
e s eep ine en n ers seg en en p in r in erse i n p in
eir g ri in ins s r res: is r ere r i
n e e seg en s in erse e s eep ine n pri ri e e r ere
x r in e seg en en p in s n e in erse i n p in s rig e
s eep ine re efine en seg en s in en e s eep ine en
n ers e e en p in ine seg en i n se r e er ine ere
seg en en p e e n e in erse i ns e een
en seg en s in r rig en p in seg en is re e r n
is p e r n in erse i n p in e seg en s in e re s ppe in

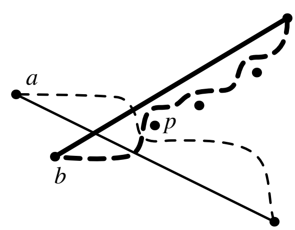
n g in is p e n s pp r g ri i i e se r inser i n n e e i n en e g ri r ns in g i e

in in e is i is s ien e e es e er n en p in ies e r e ine i is egree p n i en e seg en s re spe ifie eir en p in r in es r e er ne s p re x r in es en p in s n in erse i n p in s i is egree ree es n x r in es in erse i n p in s i is egree fi e es

tn t nt r ct n

i i ing r p i n e pre isi n e se e i i p re x r in es in erse i ns ei er in erse i ns r en p in s e n n p re x r in es en p in s n es e er n en p in is e r e gi en seg en

n e n e r e inp ine seg en s s ng s e re in n ne n n p ss er n en p in s e es s e se nn is ing is ifferen e 6 ig re i s r es e r i n n e e in erse i n p in seg en s s r g ri s nn re n r ering in erse i ns x r in es ig re s i s r es e er i e e r seg en s n p s eir in erse i n p in s r e rig s p ssi e is in erse i n p in s s p in i nesses e re i re es e e s p in in e ge e



een e seg en s e is p in in e fig re e *w ss* r e in erse i n n e se i is e firs p in er ifies ese seg en s e s ppe r eir ini i r er e e rig en p in s seg en s i ser e s i nesses i ere re n e r ier p in s in e e ge s e er in erse i n e een p ir seg en s s e ne i ness p in n er ifferen er in g i nesses s p n i p r n r e in is s nn n rep r 's s eep 6 n eir p per p ir seg en s is e *pr* r en is e i ness eir in erse i n

r t n r t t r c

n e gener se e nn pe i n rr nge en in e pre i si n e in re s n is r p e pre isi n is nee e e er ine e rien i n ri ng e r e ree seg en s A s r r e seg en in erse i n ere is i er _n e p es i seg en s n in erse i ns re ire per i ns n in erse i ns ere e i en ns n is in epen en n

e spe i se re e seg en in erse i n in i e seg en s n e re re n e i n re re r e e r ssing p irs e e pe i re re r ssings r e p e n e r i n seg en s

nge e i ns ere sing e e seg en in erse s e re s i
n nge e in erse i n r er e e p i is in e ne se i n

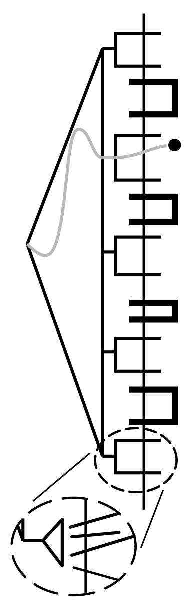
/ S S O

e rn n e spe i se re e seg en in erse i n ere e
n p e e rr nge en sing n e pre isi n es s e es ri e
e in ri n s e en s s r re n pr essing r n g ri n
e in erse i ns r rep r e rr ng en e p i i e n sis r nning
i e is si p e e s en n p ing p represen i n e
rr nge en n n e n ing egener ies

n r nt n nt

r g ri per r s s eep in ining e in ri n s in erse i ns
se i nesses re e e s eep ine e een rep r e n e r er
seg en s ng e s eep ine is nsis en i p s ing in erse i ns s
r rig s p ssi e

e e en s ring is s eep r n e en p in s
seg en s is gi es e p en i r pr i n ge
er en e nn s eep: e n pre s r e e en s
n i n i pri ri e e n e er n r
s r re n ining e seg en s r ere ng e s eep ine
e es rresp n ing re p e sin e i s se n
r er nsis en i e r i n n n n r er gi en
seg en p si i ns



n es

t tr ct r n pr c n

e s r re s res e r ere is seg en s
r ssing e s eep ine nsis s ree pie es i re i s
r e s e i in ig re

irs e gr p seg en s in re n e *u dl s* i
re e i nse i e se en es seg en s e
s e r i e sin e ere re n r ssings e een
seg en s e s e r seg en s i in n e i
re in r ere e en i e re e r e n e s res
e seg en s in s n e ree s r re n eeps
p in ers e p s n s n e s
s pp r inser e e e sp i n erge per i ns

e n e p e n es in in e is in r er n es i
ern e rs

ir e rg ni e re n es in *u dl r* n e se r ree
s pp r ing sp i n erge per i ns

e n es ri e e e en pr essing en e s eep ine re es en p in e in e e n ing egener e ses s s seg en s s ring r n ining en p in s

irs e e ng e re n es sing e n e ree e i fin ei er is e een re n es insi e ne re n e r insi e s re n es i n in er e i e re n es nsis ing seg en s en e se r n sp i e s n es n in p in en e re n e is ei er e ' en ing ' r e '

e e e ng e e n es sing e in e is g in sp i ing s e n es s e e n e is ei er e ' en ing ' r e ' A g is is ine r se r e n es in e i e erge in e ne s ep s e n rge e se r s e erging n e e n n e p si i ns re i e e n r en n es ne re n ne e re in e r ng p si i ns ese n es e in erse i ns i nesse r e p e i re e' n e R is e e en ing' r e ' n e en e er p ir in R s n in erse i n is i nesse e n rep r ese in erse i ns en e R p n p s erging R i e re n e i e i e e erging i e e n e e n rep iring e n e ree n in e is re e e i ns n es R n

in er n es re in e rre re i e p si i ns e re e se en ing p in n i nee e r ne n es r seg en s s r ing

ies e n seg en en e re seg en in s ne en s n ne s r s is p e es e pr essing

n y

e i i e s r res re i p e en e rre en e g ri rre in ins e in ri n s n ere re rre p es in erse i ns

e n sis r nning i e is n i en p in ses ns n n er ree se r es n sp i s e i n e rrie in g i e e se e erge ins rees ere re e seg en inser i n r sp i e n er erges is s is se r es n e rge

n e erges s n e n ing in erse i ns ere re per i ns e g i e e rep r ing in erse i ns n e rge p p e i gi ing g i e i e rr nge en is rep r e e p i i

n r i p e en i n e se sp rees r e n es n n e rees is n r e en n r i e n sis r sp rees e erges n sp i s se r g ri

p ct r pr nt t n t rr n nt

sing n i e r persis en s r res e n re r in erse i ns in n es sing g i e n sp e e en i ere re r i n in erse i ns i e e re sp i r erge n e e p e r e p is nge n e r ifi i ns e p is preser es ess e n e n e in erse i ns n e re r e re r ing e en p in i n es in erse i ns ere i nesse

n e e pe s r re r s ring p n r s i isi n si e n p e r n si e in g sp e i e s i s pp r ing p in i n e e p ses e rr nge en eir er in gri s ere n es s i isi n n p e r n e ges r ss e n es i e in eres ing p e is n e s r re ire i r i rs se in pp i i ns s s p ing e r n i i gr n e s r e p e r n A g r p i n es gi e n e s r re ire e nn se i r p in i n s n es r s r re represen s e r i n e rr nge en n i ne er p in is nsi ere en e e r i n nee nge e i

n r c

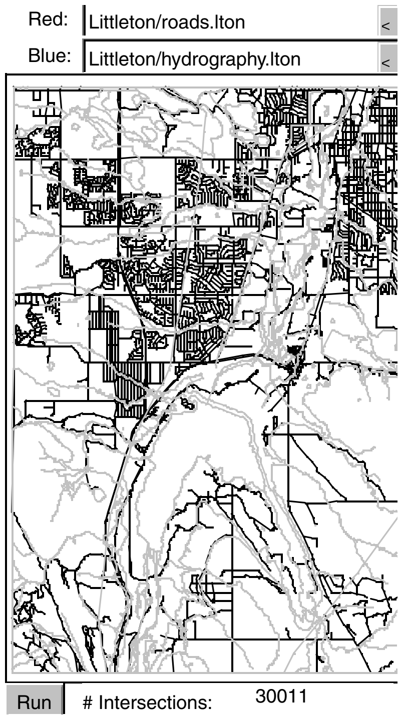
e se i ne ser i n n e e n e egener e ses n n in erse i n g ri ne p in s re e ng seg en s e i p e en er is e i i e e er re ine seg en s ese ne in erse i n p in s r e rigin seg en s s s re n se e ne p in s pri ri r isp e e isi n s epen s n e er e i p e en er r s s e ing p in ri e i r s een rne in e p s

r ppr is re seg en s n en e en p in s re repre sen e in sing e pre isi n pe ifi i n en p in ies n seg en i e n es e in e pre isi n en e n re e seg en s e n re ining egener is e se er pping seg en s i s e s r n en p in s

4 o s o

e e gi en n g ri i ing n rr nge en re e ine seg en s is p i in n n i e n sp e s in e ge r i egree i s pre i es is es i p ssi e g r n ee rre res s sing n e e pre isi n e inp r in es r ini i J i p e en i n in ig re n e n `tt \\www \ \ \`

n erse i ns in e res ing rr nge en re represen e p in ers eir seg en s s in epresen ing eir r in es re ires r i n n ers i n er rs egree ree n en in rs egree n re r e i pp sn p r n ing i e s 6 nsis en r n e p sing e pre isi n



g n erse i n r s n r gr p in r J pp e

k o s

e n e r s ing e er in erse i ns e n e in es n n ergei esp ni r is ssi ns n is g ri e gr e n e ge e s pp r r n

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T B m l m
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T y T r [2]

s a ch o o h a a oc o h o o o o c c
a a a cs. s.osa a .ac.
ad a choo o E c c Osa a s a a
a o a cs. s.osa a .ac.

A r c s d acc ac a a d so o s o o c o s d d o co o d a ass o ha o s a o o o a d acc ac ha s ach o s a co o ha s s h oss oca o s o h o . sho ho o co h o o ds o as c o s s ch as co h M o s s o co o o s d a o o s a d so o . o co h o o d o c s coo d a s s a co o a s o ha a s hos s c o s h o . E o o ds a co d a o a ca s a ad o h s s a o .

du i

T cc r cy c p r r y w c r r c rr r p rr r y r r c ﬀ c c p w r y r y ﬀ r W c r c r c rr r c w y p c r r r c rr r r pr c r c r y pr r r r c c r r ﬀ c p rr r T w c c r y y ﬀ c p rr r r cc r cy c p r

r c c p w pr c pr c c [4 y r p p y [*E o o y*[r w r p r c r r ppr pr r c p ppr c r w r w [

p p r w w cc r cy r r c pr c c w pr c c r W c p p w y p cc r cy w c p y c r r pr p c p S c p r c r c rr r c r r r r y

h s o s s o d h a d o c c s a ch o h M s o Ed ca o c c a d s o a a .

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 252-263, 2001.

r r c r r c c r c c p rr r
c pr c F r p rr r c
y r c p c w pr p
[p p r w r p w w c p cc
r cy r r pr c w c
p y r p
T r rr r w r c r r pr c r y
p w r c r y ry
r r pr w c w c - *o* c y
r w y T *o* *c* [
w c r c r pr y p r pr r
cc r cy r
T r p p r w S c c r c
r r r c S c w r c - *o*
c r S c 4 w c r r c pr w pr c
p w w c p cc r cy r
p r pr

P imi i

F r r c h (h (c c r
p r p c y y h (h (w p r
p F r r p r c pp r c
o c h (r c *o* c h (
r c t (θ t (θ r r pp r w
w r p c r θ r p c y F r r c (θ
(r c (c (= (
w r r pr r pr c T c n (p
p c r c p T
r c r c y (n

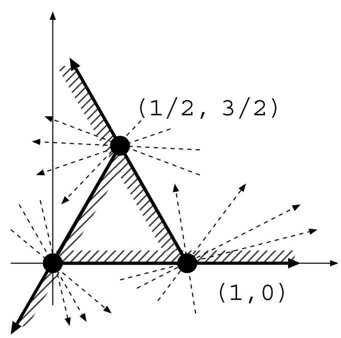
i x i

c w r c - *o* *HP* c r
p W w p r r c c
w w ﬀ r c w c r

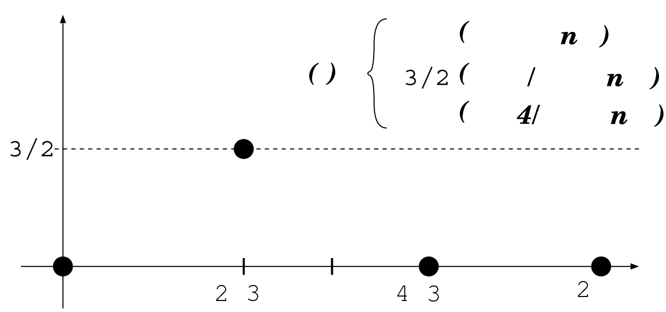
e n on 1. alf- lane e esen a on of a on e eg on
c r p r c (θ
p r pr ((θ = (θ
= <2 h ((θ

T w r pr c r y p w
r c r c p c r r c
c rr p p y c r

q <2 h $(t$ $($ θ c r y
c rr p F r p r r r F S c
r c r pr y r c r p p r c
cr c F r r
r y y pp r w F T
p r c c c w F c p r p c
c r r c
r $(\theta$ $($ θ $= t$ $($ θ r θ c
o - *o* w c y $($ θ

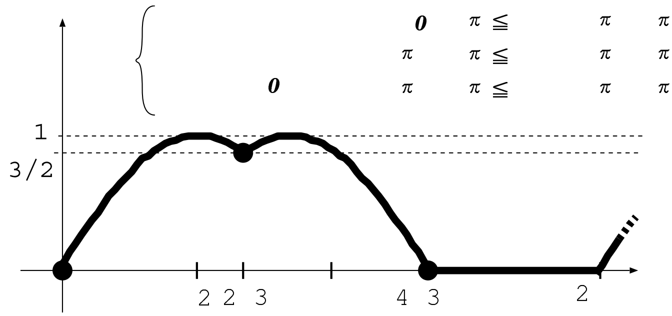


ig a a h s c o o ha a s.



ig 2 ha a s a o o a a a d sc c o .

T r c r r pp r w
pp r r pp r t $($ θ y $($ θ c θ $($ $=$ $($ θ



ig 3 ha a s a o o a a a co o s c o .

c t (θ w c p r c t (θ y ((θ c (θ (= (θ T pr p r y pp y r c p r p r

Le a . $_2$ c θ c $\theta = a$ θ c $\theta =$ θ c $\theta =$ a } θ c $\theta =$ a } c y h (h ($_2$ q h (h (h ($_2$ q h (

P oo

eo e . (θ (θ *HP* *o* *Q* *c* *y* *o* *c o* (θ (θ } *HP* *o* *Q*

P oo *Q* q $_{<2}$ h ((θ $_{<2}$ h ((θ c rr p $_{<2}$ (h ((θ h ((θ c y $_{<2}$ (h ((} θ *Q*

eo e . *o co* *o* *Q* *n* (*Q* *o* *c o* (θ (*Q* θ } *o* *HP* *o*

P oo S c c *Q* h (t (θ q h (t (θ h (t (*Q* θ c y (θ (*Q* θ } (θ

F r r *Q* w *Q* y *Q* $q|$ } c w c r c w c r pr y

eo e . *o co* *o* *Q* *o* *HP* *o* *Q* (θ (*Q* θ

P oo F r c r w p q y p r c w

$$(\quad q\ \theta = (\quad \theta \quad (q\ \theta \qquad (
$$

c r q Fr T r 4 P (q θ } (q θ
w c q P (θ (q θ } y (c (θ P (θ }
w (q θ = (θ (q θ S r y w c w (Q θ =
(θ (Q θ

F r r Q w ff r c Q y $\ominus Q$
$q|$ } W c pr r r w ff r c

eo e . *o co o* Q *c o* (θ (Q θ
HP o $\ominus Q$

P oo Fr r ($\ominus q$ θ = (θ (q θ w r q
r p w c pr r y r r T r

W p r c r c c w w
F

E 1 T r p (
T pp r t (θ w c p r y θ
c $\theta = a$ θ c θ c (θ $= a$ θ c θ

E 2 T r c rc r r w c r (
r pp r t (θ c c p (a c (θ
(θ T r w

$$(\quad \theta \;= (a \quad c \;(\theta \qquad \theta \;(\qquad (\theta \qquad c \;\theta$$
$$= a \quad \theta \quad c \;\theta$$

E T c p n
Fr T r 4 w (n (θ = n (θ } (a
c r (n (θ pr w

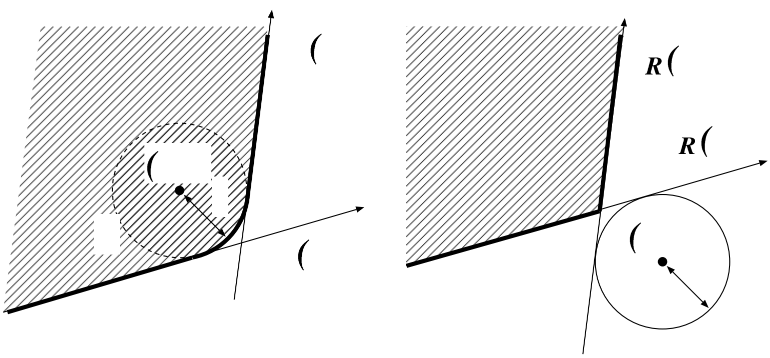
$$(\; n \; (\quad \theta \;= \; {}_{n} \; a \quad \theta \quad c \;\; \theta\} \qquad ($$

r r F q

.1 eg on ons on f o alf- lane e esen a on

c w ry r pr c p y
c r c r (w c r r c rr p
c r
p p r w c r y c r r
c rc y p r T c r r
p rp c c r pr r y c r c rc p (w c
c rc r r c p y (w c c r c
r c w ff r c
c c c r c p c
y c r r (θ $= a$ θ c θ w c w c

HP (r c rc r w
c r (*a* c c r c c rr p c r r c
c r c r $_\alpha$ $_\beta h$ ((*a* θ c θ θ w c w c
o W r y w $\leq$ c
> w c c r c pr r w p
r $\theta \leq$ (r r ($\theta \leq$
T p pr r c cc r c
r rr y w c rc r
rc (c F 4 c r
rr y w (r c F 4



ig o s o o s.

c c rr p c r c c c
r c y r c w yp pr r S c p
r r cr r r c r c c
(w r r pr

eo e . *co o co o o HP co o o*
c o HP c co c (

A u u d u i

c w w cc r cy r r c pr
c c w pr c c r rr r r r c r
pr c p w c p y r
p T rr r pr r r pr y W
r c w c w c pp y
r c p r p r [

.1 n o el

-*Po* c r r pr p c p
w c p w c r r pr c w p
cc r cy p y T y r w
ff r p r c c r W (q r

y r pr p p T r
w

e n on oss le se s of o n s . n
T p q q_n o o o o r
c $\to T$ ($= q$ ff q W c c q o o
o (p p

c p c r cc py c y c y
p p w c r rr r
o o c o c r r r c c c r
p p

. on e ll of o n s

(p c (=
n (| (} r r $\underline{\ n\ }$(y C C (S
r c p c $\overline{\ n\ }$(y C C (S
p c c y p c
$\underline{\ n\ }$($\overline{\ n\ }$($\underline{\ n\ }$($\overline{\ n\ }$(rr r
y p c r r pr
w r [

eo e 9. c o S (θ } HP o $\underline{\ n\ }$(

P oo W r r y n F r r

$$\underline{\ n\ }(\quad = \quad = \quad n \bigcup_{=}^{n} \quad \} \qquad ($$
C C (S

Fr T r T r 4 (θ $\underline{\ n\ }$(y

$$(\theta \ = \quad _n \ (\quad \theta \}\} \qquad (4$$

S c n r p c r w c c r r
(4 (θ } q (θ c
h ((θ } θ c rr p h (t (θ c (θ c
p n (θ }

eo e 1 . c o S (θ } o HP o $\overline{\ n\ }$(

P oo W r r y $_n$ S c $\overline{n}$ (c w
pr [pr w

$$\overline{n} (\quad = \quad n \; (\overline{n} (\quad = \quad n \quad \bigcup \quad n \quad \bigcup_{=}^{n} \quad \} \qquad ($$

Fr T r 4 r $(\theta$ $\overline{n}$ (y

$$(\theta \; = \qquad _n \; (\quad \theta \; \}\} \qquad ($$

S c $_n$ r p c r w c c r r
((θ } q (θ c p
h ((θ } c rr p h $(t$ (θ c $(\theta$ c p
$_n$ (θ }

W r r r c T r 9 T r y $\underline{n}$ (θ
$\overline{n}$ (θ r p c y

o olla y 11. *o o -Po y co o y o n*
c o y o $\underline{n}$ ($\overline{n}$ (*c co c* $(n$ n
c -Po y n c c

P oo c $\underline{n}$ (θ $\overline{n}$ (θ r p r
c p p r c c w c
y c p y r c p r $(n$ $(n$ (y r r
[$(n$ r p c y c c c r c $(n$ n
[r $\underline{n}$ ($\overline{n}$ (c c r c
$(n$ $(n$ $(n$ r p c y
c r y n c rc c p y $(n$
c c r c $(n$ n c $\underline{n}$ ($\overline{n}$ (c
c r c $(n$

cc r cy r c w
c p r p r T r 9 w
$\underline{n}$ (θ $\underline{n}$ $(W \theta$ w c c r c $\underline{n}$ ($W \theta$ y c p
$\underline{n}$ (θ $\underline{n}$ $(W \theta$ } w c $\underline{n}$ ($\underline{n}$ $(W$
c r y $\underline{n}$ (W (r p c r c W
p p r c r Tr c rr p
ry w c rr r

. nko sk of on e olygons

c p y w r c r p c y
Q w W W (Q p w
Q y (Q = Q | (Q $(W$ } w r
w w p c

Q p p y pr r c y
w
$\underline{Q}$ $S(P$ r c p w
$\overline{Q}$ $S(P$ p w
c pr $\underline{Q}$ $\overline{Q}$ rr r
Q

Le a 1 . *co o y o c c y -Po Q o W o o o o* $\overline{Q}$ *co*

P oo r $\overline{Q}$ q $\overline{n}($ $\overline{n}(W$ Fr c y $\overline{n}($ $\overline{n}(W$ (r [$\overline{Q}$ c

eo e 1 . *co o y o c c y -Po Q o W c o* $(\underline{n}(\ \theta$ $(\underline{n}(W\ \theta$ $(\overline{n}(\ \theta$ $(\overline{n}(W\ \theta$ *HP o* $\underline{Q}$ $\overline{Q}$ *c y*

P oo y r $\overline{Q}$ Fr T r T r 4 T r $(\theta$ $\underline{Q}$ c c w

$$
\begin{aligned}
(\theta &= \quad S(P \quad (\quad \theta \} \\
&= \quad P \ C \ (\quad C \ (\quad (\quad \theta \quad (Q \ \theta \} \\
&= \quad P \ C \ (\quad (\quad \theta \} \quad C \ (\quad (Q \ \theta \} \\
&= (\underline{n}(\quad \theta \quad (\underline{n}(W \ \theta
\end{aligned}
$$

S r y w $(\ \overline{Q}\ \theta = (\overline{n}(\ \theta \quad (\overline{n}(W\ \theta$

o olla y 1 . *co o y o c c y -Po Q o W o HP* $(\underline{n}(\ \theta$ $(\underline{n}(W\ \theta$ $(\overline{n}(\ \theta$ $(\overline{n}(W\ \theta$ *o co c y o o* $\underline{Q}$ $\overline{Q}$ *c co c* ((*c y*

. a e e of o n s

y pp y r c p r p r [r $am($ c r c r pr y $am(= \ _{<2} \ (\ \theta \}$ w r $(\ \theta$ $(\ \theta$ $(\ (\ \theta$ c w $\underline{t}(\ \theta$ $t(\ \theta$ $\underline{th}(\ \theta$ $C\ C\ (S\ (\ \theta \}$ $\overline{th}(\ \theta$ $C\ C\ (S\ (\ \theta \}$ rr r r c c p r y

eo e 1 . $\underline{am}($ $\overline{am}($ *o o o o :*

$$
\overline{am}(\quad = \sum_{<2} \overline{th}(\quad \theta \} \quad \underline{am}(\quad = \sum_{<2} \underline{th}(\quad \theta \} \qquad (
$$

P oo T r q y r W pr c q
_(*q* p c w *q*
(*q* } W y *S* _(*q* }
am(q c *am*(c r c
r r w c c pr y c r c r
r
T r w pr <2 *th*(*θ* } = F r r
c c *th*(*θ* r [c rr p
p c w W pr *th*(*θ*
q w c r c y w r p c
* (* = *th*(=
c _(= p p
y (= _(W y pp r
p rp c r y *w* r pp r r y
W w *h* (y c r c
h (p p c
(= _(c cr r p w *w*
(r r (_(r r _(
w c c r c S r y w c w *h* (*w*
y r r p p w r p p
c p w *w* T c * (* =
th(=

r r c r r c [9 pp p r
y 2 *th*(*θ* } ff r r
r c y c rc Fr r y r c
r pr p r

eo e 1 . *th*(*θ* = *n* (*θ* *n* (*θ* }

P oo W y (*θ* c *n* (*θ* *S* (*θ* }
y (*θ* c *n* (*θ* *S* (*θ* } F r
((r p c y
y r pp r pp r
r r pp r w
r y pp r

h (= *h* (= (

c *th*(r y
c *h* (((r p r y T
th(q c w ((
c *h* (((y *h* (*h* (
w c p *th*(=

th(= q c
r r [r ppr c r
o T pr q c p 2 th(θ } p
w J r T pr y
o c p r y c c c [r
y w pr c y p c r c
w w (k (
(k

eo e 1 . *o n co o -Po n c-*
o (θ = n (θ } 2(θ = (2 n (θ }
(θ → N (θ = | (θ = (θ }
th(θ *y o o q o :*

$$\overline{th}(\theta) = \begin{cases} (\theta \quad (\theta \\ (\quad (\theta = (\theta \quad \vee | \quad (\theta | \\ (\theta \quad {}_2(\theta \quad (\theta \quad {}_2(\theta \quad \} \\ (\quad (\theta = (\theta \quad | \quad (\theta | = \end{cases}$$

P oo r *th*(θ r y w c
Fr r *th*(θ = n ≠ (θ (θ }
th(θ c

o olla y 1 . *o o -Po y co o y o n*
c am(*c co* (*n n am*(*c co* -
c (*n* (*n n o c*
-Po y n c c

P oo W c c p *am*(c w *n* (θ c c
r c (*n n* T c p *am*(w *n* (θ
rr r c c
p p c (*n* (*n n* [

. e o le s

c c w c pp y r c p r p r
c pp r w c c p r c c p r w
c p r w c c r *n* (
n (cc r cy r c p r
F r p c r c p c *t*(*Q* r c
p y *Q* T c y <2 (*Q* θ (θ }
(r *t*(*Q* r c *Q* c
p c w c p y w r c r
p c y *T U* y <2 *n* (*U* θ
n (*T* θ } S r y r pr c cc r cy r
r y pp y r c p r p r w r w c r w

p c p r r r 2 $\underline{\quad th}(\theta$ $\underline{\quad th}(\theta$ } w r r c r c w c r $\underline{\quad th}($ $\underline{\quad th}($ r r r

u i

p p r w c r r c pr c c r p p p r w y p cc r cy T w r - *o* r c p r p r T pr r cc r cy r r y rr r pr p p r c c p c y c pr c T y r w p c ﬀ r rr r

1. . . a a M.d .Ma o s a d O. ch a o o s c s a s a d h O d o o o a a s *oc n s o t nn CM pos on Co p t t on o t* a s 1 .
. M. a ah a d . a a Effic o h s o o o a s sa s *In o t on oc ss n tt s* o . a s 1 1 .
. . a .M. o s o a o a acc a o o *Int n t on Jo n o Co p t t on o t pp c t ons* o . o. a s 1 1 .
. . . a c osa . a sso . a os M. a a o co h a o h o o s h a o a o os o s *Int n t on Jo n o Co p t t on o t pp c t ons* o . o. a s 1 1 1 .
. . as . a . o E s o o d o s a o h o c s co a o s *oc n s o t nn CM pos on Co p t t on o t* a s 0 1 1 .
. . as . a . o o s c o o o a s h acc a s *o t c* o . a s 0 1 .
. . sh d h E o o *n* s *n o n In o t on oc ss n tt s* o . a s 1 1 1 .
. . a a . as o a d . o a o o h c s *ct ot s n Co p t c nc* o . 1 a s 0 1 1 .
. a d a a o co h a o h o d scs a d a ca o s *Co p t t on o t o n pp c t ons* o . 1 o. a s 1 1 1 1 .
10. . o a d . o ssa o a o a o a d ac oca o *o c n s o Int n t on Con nc on p t ons s c n M n nt c nc* a s 1 1 1 0.
11. . o ssa o o c o s h h o a a s *oc n s o I M C 3* a s 10.0 1 1 .

h s a c as oc ss d s h E ac o ac a h s

T g c s ms, h g P P s, d h H phs

A s ir

ar a a cs Osaka k rs
4 98 1 sa a ka as wara Osaka 8 8 8 a a

bs ac *gl* s a r cal c la c s s r r a lar sks s_1 s_n a r a l r s s A B a s c a r r ac s c c s w a r r A B r a s s s g l l s a r cal c la c s s a r r z al s s a_1 a_p a r cal s s $_1$ a r c a l r s s A B a D w A r z al s c a r ac a c c s w a r r B D r s a a r ac c c s w a r r A r s a r a w ca c s r c a la r a la a a la q a ra la acc r a ac c l s a r s c l s a r w a s r r r a l c ac s s s r al la ar s a r ra s c s r c r

O O

C nsi er se $\mathcal{S}$ S S ir r is s rr nge in e p ne in i n ir r is s in erse s ne p in A r ing e en e ir r is s e n ns r e gr p se er e se rresp n s $\mathcal{S}$ is e s see n e r n in e p ne i e ge r ssing r n r i r ri gi en gr p n e p ne es ere e is s n rr nge en $\mathcal{S}$ is s s is pr e s een r i e s e in 6 ere re se er rese r es n erning e pr e e er gi en gr p n e p ne n e represen e s ge e ri e rr nge in e p ne n is p per e s nsi er s pr e

e e *s* *pl* gr p i e i ps n ip e e ges i is *dd* *d* in e p ne s n e ges n' in erse e ep eir en s gr p is e *pl* gr p e en e e er e se e e ge se n e e se n respe i e A er e *v* r n e ge *e* p ne gr p is s i e *u* *r* resp *r* i *v* r *e* is resp is n in i en e infini e regi n A *y* *l* e ns e eng e A p ne *r* *ul* is p ne gr p i e e n e e i e p ne *u* *dr* *ul* is p ne gr p i e e n e e is e s see e er r ng i n n e p ne is *p* *r* i e i s er i es

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 264-273, 2001.

n e re n ies n en er i es e ifferen rs e e n i p ne ri ng i ns n p ne r ng i ns n is p per e in r e ge e ri e s rr nge in e p ne e *r l sys s* n *r h l pl p r s* rresp n ing p ne ri ng i ns n p ne r ng i ns n es ri e i s re i ns r gr p e re i p in ie e e re een e in e p pers in epen en is p per p s r e p ese res s ge er n r er ser i n r e

o y o

e e ri ng e in e p ne i er ree s r ig seg en s n pp se ri ng r is s re p e in e in eri r s e er e e er in i es i n inner p in seg en s e i r n inner p in s e n is se e s *h s* r *h s* e en s n rr nge en is s i e *r l sys* e en reg r e se ri ng e n s s e e reg r ri ng e n s s e s s e s e i e e e s e in ri s r re i respe e en

A p r i rien e p ne gr p i er e se is s i e e *h r ph* i ere e is s i e i n :
s

i r n r
ii s e ree n n rien e e ges r ing e er ri ng e n r n x s n rien e e ges x i n n i x i s

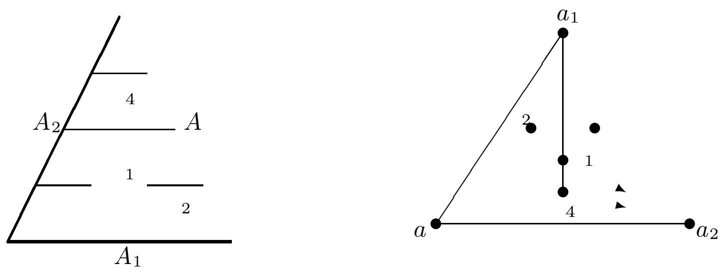
r iti n r y i M nd z nd n ti) *h h r ph r l sys s pl r ul wh h h r d s r d s h ry r r h s u d r ly r pl s 1*

e n rien i n p ne ri ng i n i e inner er e e gree e n *r r* A p ne ri ng i n ss i e i n inner rien i n is e n *r r d pl r ul*

e i gr p is p ne gr p i e er n n rien e e ges n e inner rien e e ges sin e e inner er e s egree e en e s e 6 e ges n e er n e n er's r e er p ne gr p s s 6 e ges n e e i s en is ri ng i n s is n inner rien e p ne ri ng i n

e n erse r p si i n s s s in e ing e re

r r y i M nd z nd n ti) *r ry r l sys* $\mathcal{S}$ *h r s s u u r r d pl r ul*



g 1 r a l c ac s s a s ra

wh h s h h r ph $\mathcal{S}$ C rs ly r ry r r d pl r ul h r s s u u r l sys $\mathcal{S}$ wh s r l r r ph s

e gi e s e pr e er p r sin e e firs p r s een es rie in r p si i n

pp se n inner rien e p ne ri ng i n s er er i es c n inner er i es e e e inner er i es v v v s s e x en e n e ge rien e r x r is s e ns x is rien e e e p r i rien e p ne gr p ine r re ing c r e s essi e efine e s gr p s s e ser e e ing i

Cl e R e e regi n n e n ne ess r rien e se W eng en R s e e ges rien e r er i es n W r e in eri r R

er's r e n n e i ere re er i es in e in eri r R en R s e inner e ges in e e inner er e in R s egree ere re rien e e ges in ing r W s e i s

pp se s n r s s ere s n s in e e er i es n e n g ing e ges e e s s n s s en e ere e is s er e s i $\leq$ $\leq$ s s s n s s e v e s s n e v r e e ges in i en v in e ep v s n v s re rien e v C e r e s pre ise er i es

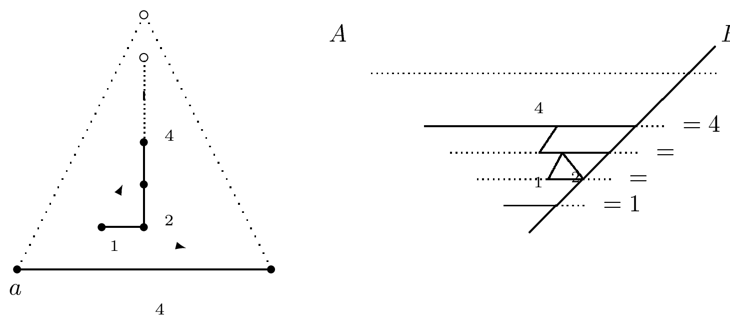
e ns r ri ng e n s s e $\mathcal{S}$ i er ree seg en s C rresp n ing c n ri ng r is s rresp n ing v v respe i e Dr ines : x n : x in x R : x r e rigin ese ines n rresp n e er i es n respe i e C e r is rresp n s

pp se e er e v s neig rs v_{s_1} v_{s_t} in in e ise r er ere $v v_{s_1}$ $v v_{s_t}$ n v_s v r n v is rien e v in

pp se e rr nge en $\mathcal{S}$ $_1$ nsis ing s re een ns r e ri ng r is s s n in ig ns r $\mathcal{S}$ s

i ne seg en is n ine
ii en s re n S n S i rresp n s n s
iii e ir er e e e p n n S n S s r in e ere v is rien e v in i v

in e e ps S S e r in e e n e ps s_1 n s_t e r in e s ri gre er n e n p e s in ii in p ri n seg en C n ine rresp n ing e er er e c



g s r c r

r ry *L h s ll r l sys s L h s ll r r d pl r ul s h h r s s* : *su h h r y s h h r ph*

r r y i M nd z nd n ti) *ry pl r ul h s r r*

e rs in e pr e e re ns r i n ere is n er pr e re in i n n e n er er i es e e p ne ri ng i n n e n inner rien i n s n rien e e C en e n in n er inner rien i n en e nging e ire i n e e ges C e is per i n *y l r rs l* A i n re ifferen inner rien i ns e s e p ne ri ng i n en e n efine e *su r* e e sp nning rien e s gr p s n x : x en r n er e v

e egree v in e s i s in egree in sin e e er e s e s e egree in n en e is n e ge is in ni n se er e er rs is e ns n n e r ns r e in e er repe e pp i i n e re ers s ere e i e nsi er e er r n n n e r ns r e in e er e re ers s n es A e re ers e eng is e n y l r rs l

r t t nd n) *L d r r s h s pl r ul w h r s h d r u l h h r*

n r er pr e e re i s es s re ers n e C i in n inner rien e p ne ri ng i n n e ine se en e n e re ers s

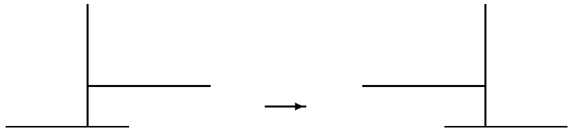
sing er's r e n n e ere is n rien e p r g e in eri r C r x C C s s

e R e e e regi n n e C C i e v e er e n C i s n rien e e ge r e in eri r C C nsi er rien e p s r v p ssing r g n inner rien e e ges R n R ng s rien e p s r v ere is s n rien e p re ing er e n C er n v en e re ne n en e e s pp se ere is n s p s e er er i es R ie n e s e regi n s R ng e se er regi ns ine r R ing C e r R is n e n ne ess ri rien e se eng s ri gre er n R s n in ing e ge r e in eri r r e n r is n r i s C i n en e ere e is s re ire

en n ne rien e p s n C e een x n r n rien e e n ing ess n er es n C in i n p esis e rien e e n e re erse An e re erse n e n er rien e p n C e een x n s r n rien e e n ing ess n er es n C is n e re erse si i r n e res ing rien e gr p n e ire i n C s een re erse sin e e rien e p s re erse i e ere re e re ne

e re rifies e re i n e een ri ng e n s s e s n inner rien e p ne ri ng i ns e re is re i n n e r ns e in e re i n e een s e e i en e ss ri ng e n s s e s n n n rien e p ne ri ng i ns s s

e n e ri ng e n s s e s pp se eir i gr p s n ′ i re inner rien e p ne ri ng i ns r p si i n n e r ns r e in e er re ersing e in is e ns s ree ri ng r is s n s i respe i e n nging e en s s n in ig fi ing e en er ri ng r is s r ns r s in e e n' n e er n n is pi e r ns r e in e er e s ri ng e n s s e s n re u l n en e i e n e r ns r e in e er se en e e e e r i ns ng ree ri ng r is s



g a a ac c a

e e n n rien e p ne ri ng i n n e n inner rien i n is e i gr p ri ng e n s s e en is s i e *h r d r ph*

r ry *L h s ll r l sys s L h s ll r d pl r ul s h h r s s c* : *su h h r y c s h h r d r ph*

nsi er n e re ers s re nee e r ns r inner rien i ns p ne ri ng i n e e p ne ri ng i n n e n e inner rien i ns e e e ini n er e re ers s nee e r ns r n e e e ini ere n r nges inner rien i ns

r t t nd n)

$$\leq - \quad —$$

r iti n t t nd n) *h r s s pl r ul w h w r r s d su h h*

$$\begin{cases} - \quad — \\ - \quad — \quad \text{er ise} \end{cases}$$

e re n r p si i n e r er nn e i pr e e n e re e e ien e pper n is –

O O O O

e s in r e ge e ri e e n *r h l pl p r* i s re i n p ne r ng i n ss i e i si i r p r i rien i n n res s r ese e s p r e ri ng e n s s e s n eir rresp n ing p r i rien i ns

An *r h l pl p r* C is ge e ri e n e p ne nsis ing fini e n er ri n seg en s n er i seg en s in re ng e R n e r seg en s C n i ri n s e er en p in e in i es i n inner p in ei er r s e n e er en p in e in i es i n inner p in ei er C r s e r seg en s n i n en p in in i es i n inner p in en e seg en is s i *h* ser e r n n esn' i n esn' i e reg r r g n p ne p r i i ns n s e s e i e e e s e e en seg en s i in es i e i n e een e se re ng es n

ere e is s i e i n e een e se re ng es n

A p r i rien e p ne gr p i er e se c x x is s i e e *h r ph* i ere e is s i e i n ϕ : s

i ϕ ϕ ϕ c C ϕ x r n r q

ii s e r n n rien e e ges c c r ing e er ri er n r n x s n rien e e ge x i n n i x i s

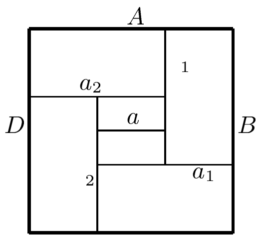
iii i s e n e e x x ere x x c x x n en ere is re ng e i is n e seg en s ϕ x ϕ ϕ x ϕ in is r er n se in eri r is e p

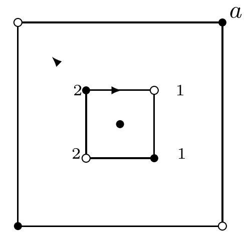
e ir n i i n is nee e g e rresp n ing n i i n r ri ng e n s s e s s n nee e is n n nne e p n r gr p is ni e e e e in e p ne A p ne ri ng i n is s nne e i e p ne r ng i n is n ne ess ri nne e n ig e ifferen e e ings s i is i p r n ne spe ifie e e ing s e p n r ip r i e gr p rresp n s fi e r g n p ne p r i i n

e i gr p is p ne r ng i n in i e inner e ge is rien e s e er inner er e s egree e e is rien i n n *r r* r e p e see ig A p ne ri ng i n ss i e i n inner rien i n is e n *r r d pl r ul* e ing e re s een pr e

r t nd t n) *r ry r h l pl p r h r s s u u r r d pl u dr ul wh h s h h r ph C rs ly r ry r r d pl u dr u l h r s s u u r h l pl p r wh s h r ph s*

r ry *L h s ll r h l pl p r s L* Σ *h s ll r r d pl u dr ul s h h r s s* : Σ *su h h r y s h h r ph*





g Or al la ar a s ra

r 6 t nd t n 8) *ry pl u dr ul h s r r*

e e p ne r ng i n n e n inner rien i n App ing e re ers e n in ifferen inner rien i n i i r e se r p ne ri ng i ns e n in e ing e re

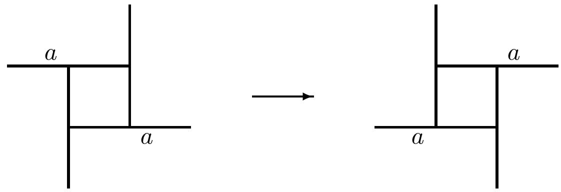
r t nd t n 8) *L d r r s h s pl u dr ul h d r u l h h r*

e re 6 rifies e re i n e een r g n p ne p r i i ns n inner rien i ns p ne r ng i ns

e n e r g n p ne p r i i ns pp se eir inner rien i ns n ′ n e r ns r e in e er ne e re ers nging x x l in $l x$ x is e ns s inner seg en s l s l i l respe i e n nging e en l s s n in ig fi ing e en er ri n g r is s r ns r s in i e e n' n e er n n is pi e r ns r e in e er e s ri ng e n s s e s n re *u l* n en e i e n e r ns r e in e er se en e e e e r i ns ng r seg en s e e n n rien e p ne r ng i n n e e n inner rien i n is e i gr p n r g n p ne p r i i n en is s i e *h r d r ph*

e re s 6 n e e i en e r g n p ne p r i i ns e re i n e een n Σ in C r r n e r ns e in e re i n e een e e i en e sses r g n p ne p r i i ns n p ne r ng i ns

r ry *L h s ll r h l pl p r s L h s ll r d pl u dr ul s h h r s s c : su h h r y c s h h r d r ph*



g a a ac c *a* *a* a

4 k

n is p per e en i ne e res s n ri ng e n s s e s n r g n p ne p r i i ns An e es ri e ese rresp n s inner rien i ns p ne ri ng i ns n inner rien i ns p ne r n g i ns n e er p ne ri ng i n n e er p ne r ng i n e n inner rien i n n n inner rien i n respe i e

ser e e er p ne gr p n e r ns r e in ri ng i n ining er e e n n ri ng r e s e r s n rien i n s e inner er e s egree s g e egree r e er e nn e spe ifie n gener ne ess r n s ien n i i n r gr p e n rien i n egree spe ifie r e er e is n n s in e ing pr p si i n

r iti n r n) *L r ph d l* :
u su h h r h v $\leq$ *v* $\leq$ eg *v h h s r su h h h r v h s u d r v d ly r y S*

$$\sum v \quad S$$

wh r S d s h su r ph du d y S

e re s n 6 n e pr e r p si i n s s e s s p ne ri ng i n s n rien i n s e inner er e s egree e n e er er e s egree e C nsi er n S n e s gr p S in e S er's r e e S $\leq$ S 6 sin e S is p ne gr p n e er n e e $\sum$ v S S 6 sin e S n ins s er er i es s r p si i n s re ire rien i n in re ing e rien i n r e er e ges e n in n inner rien i n e n si i r pr e e re 6 sing e e er ip r i e p ne gr p er i es s s e ges

in e e er p ne gr p is n in e s gr p s e p ne ri ng i n s es ri e e n e represen e s rr nge en ri ng e

n s s e e er is represen i n ig e *r* er e ri
ng r is i e er e n ing ers sin e i s een ine r
ri ng e n s s e re ing se er ri ng r is s n er e ges
e rn en i n p ne ip r i e gr p s er p ne ip r i e gr p
is n in e s gr p s e p ne r ng i n en e s n ri
en i n i e inner er e egree s g e egree r
e er e nn e spe ifie n n e represen e s s rr ng en
n r g n p ne p r i i n i ig e ree en s se er seg en s
en e n nsi er ge e ri rr nge en i e s e p g ns
n i n ree er e n seg en s i e n r g n p ne p r i i n n
ri ng es i e ri ng e n s s e re p ssi e ere is n rr nge en
nsis ing g ns in e in eri r g n i n ree er e
en e rresp n ing p r i rien e p ne gr p s e er i es
n e ges e er p ne gr p i er i es e
s 6 e ges s s epen n n e nn fi

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O O

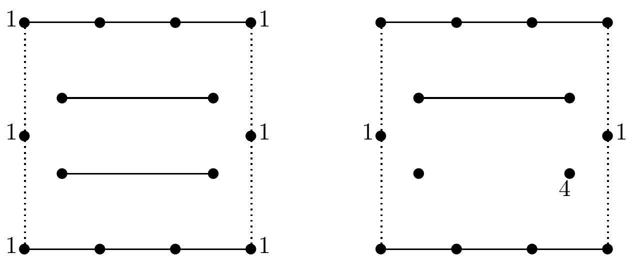
A *u d r ul* n se s r e is si p e gr p e e e n s e e is n e e eng A *d l sl d* in is rep e n e ge i r in e g n regi n i e *d l r* r n er e *v* egree is rep e e ges *v* n *v* i *v* n *v* in ri er regi n e ese in s e r i ns *d l ps* e i e r ng i ns n re s i e *u l u d r d l ps* i e n e r ns r e in e er p e rp is se en e i g n ips ere e een s ies –6 n e e i en e er r ng i ns n se s r es n er i g n ips e er ere s ne er een n p per ri en n e re i ns ip e een i g n ips n rings r ng i ns n se s r es

A *l r* gr p is n ssign en : rs er i es s *v* r e e ge *v* n is e *l r* i ≤ e *hr u r* is efine s e ini s i s ring C nsi er r ng i n n se s r e i is re s i e ring : n e si i n r n n e ge en i e e e en nn e se s ring e r ng i n ine r i g n s i e en i i s ring i is r p ssi e ge i n e r ing ne ig n e pe e nne i n e een i g n ips n e r i n ers r ng i ns n se s r es e er e s s s rprising s s A gr p e e e n se s r e is s i e *r pr s* i n essen i si p e se r e n ee s in e s p in s e i s is represen i e is e e *r pr s y* ee r e e i s represen i i

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 274-279, 2001.

r *r y l s d sur h r s s ur l u r R R su h h w R r pr s u dr ul s d r u l u d r d l ps h*

ig re presen s represen i e r ng i ns n e r s e i n e ine r e re ng e i en i ing p irs i s p r e si es is e r e n e r ns r e in e er sing e i g n ip e er e e ne s r i n er i e e rig ne s s e n ers er i es s gges s e ss p i n n represen i i in e e e re is nee e



g 1 w q a ra la s r s

e in i en e r i n ers es n i p e e i en e r ng i ns n er i g n ips in gener e en i e e s ien rge represen i i e er i e nfine e s r es e rien e en e n pr e e ing e re :

r *r h r l l s d sur* S_g *us h r s s ur l u r R R su h h w R r pr s u dr ul s d* S_g *r u l u d r d l ps d ly*

n e er n e s e s e en r n n rien e se s r es s e es n e n pr e n e ing e re r e :

r *r h r l l s d sur* N_ε *us* ε *w h* ε *r h r s s ur l u r R R* ε *su h h w R r pr s u dr ul s d* N_ε *r u l u d r d l ps r* ε

i g n ips n e r i n ers n e s i e in ri e s e er e s in r e n ge r i in ri n e e e p ri in e i n e e p ri i e i e e een se in ri e s in r pr s e e re s i i e gi en in e i n ese e re s s e er s eri s r e re er es n n e eep res s in n

y

e e p ri s efine firs in r s ies n e e i en e er
r ng i ns n er i g n ips n s een e e pe in r s ies n
rings r ng i ns e s re ie e e r e p ri ies gi en in
e er ere e en e e n en gr p n e i ensi n
g gr ps i e ien R ; R ss ing e re er n s
se si e s in ge r i p g
e e se s r e e n rp is : ***Z***
y l p r y er A se r e is s i e r *dd* n er e
p ri i r ere en es e p ss i
e ngs n p r i r e p ri is s i e *r l* i r
e p ri ies n re s i e *ru* i ere is e rp is
: i in es n rp is : i

A gr p e e e n se s r e is s i e *ly* *dd d*
n i e e is n pen e n is n e se e en
eng is e s see se s in n e en e e e gr p
e e s e eng i e re pi e er n n
se s in gener e s e p ri n e efine
W W r e se W in ere W s n s r e
eng W e is e *y l p r y* e e p ri ies n ,
gr p s n re ngr en i ere is e rp is :
i h) , is e r is ri i i n n i is ip r i e
e : ; ***Z*** n : ; ***Z*** ; ***Z*** e e n ni
re i ns An e p ri : ***Z*** rs r g ; ***Z*** n
; ***Z*** n r is ere re rp is s : ; ***Z***
Z n : ; ***Z*** ***Z*** s n s n n
e e er ine ni e e re i n : ; ***Z*** ***Z*** i e ngs
e ***Z*** g gr p ; ***Z*** r e pre er e
r e e p ri r er n en e is ss e er r g
e S_g en e e rien e se s r e gen s i n e
ine s e sp ere i n es C se p irs si p e se r es
n S_g e rresp n ing ne e n es s n
r ss e n e r ns erse r n e n r sses
n er n i e s s e g g presen s se
S_g; ***Z*** n n e p ri n s n n e e er ine
g g s e en e
g g i is se en e 's n 's

r *A y r l y l p r y* S_g *w h s ru*

e N en e e n n rien e se s r e gen s i n e
ine r e sp ere ing r ss ps e er e s ssi e
n n rien e se s r es in sses epen ing n e p ri eir

gener e pe N n e reg r e s e rien e se s r e S i ne r ss p e e se si e si p e se r e x ng e r ss p n p irs si p e se r es n S s e s in e pre i s s x r sses n n n en e s s e x r s se N ; $\boldsymbol{Z}$ n x is ni e e e en r er in N ; $\boldsymbol{Z}$ e e press e p ri s x

r *A y r l y l p r y N w h s ru ly d C*

e e en pe N n e sep r e in ne K ein e n e ri en e se s r e S C se p ir si p e se r es n e K ein e s is si e n s pen e K ein e in n nn s n r sses r ns erse e n e en e e s s e si p e se r es n N is r s se N ; $\boldsymbol{Z}$ n is ni e e e en r er in N ; $\boldsymbol{Z}$ e ri e

r 6 *A y r l y l p r y N w h s ru ly d r h l l N s ru h r r*

A e p ri er N is s i e *yp A B C* r i i is ngr en e e e p ri ies in e re s n 6

OO O O

e s pr e e re s n r g is se i n in r ing s e n n res s e nee in r er

e ing e re presen s n en res in e e r i g n ips in r ng i ns n se s r es pr e in is i p ies e e p ri is p e e in ri n r e e i en e er r ng i ns i s ien rge represen i i

r t) *r y l s d sur h r s s ur l u r su h h w u dr ul s d w h r u l u d r d l ps d ly h y h ru y l p r s*

n e er n ins n s pr e ere e is s n r n er s e er represen i e r ng i n n S_g is r e s s r ng i n is ei er ip r i e r r i in e ere re n ngr en e sses e p ri ies er n rien e se s r e ese ern i es s rresp n ri i e p ri n n n ri i e p ri respe i e s e re s i R

S_g sin e n R represen i e gr p s e s R er i es e re s r i

r n n rien e se s r es e nee e ing e re i r eri es e r i n ers r ng i ns i s ien rge rep resen i i in er s e p ri ies:

r 8 t i nd t) *h r s s ur l u r su h h y r pr s u dr ul* N *s l r l d*

i *s hr d ly s r l*

ii *s hr d ly s h r yp* C *r*

iii *s hr d ly s h r yp* A B *r*

in e e i en r ng i ns e ngr en e p ri ies e re s i R N e re n represen i e r ng i n n N s e p ri pe C i i is r i i e s r ng i n n N s e p ri pe i i is r i is i p ies ε represen i e r ng i ns n e ngr en e p ri ies i ε r er re i e re R represen i e i R N en e e e s N er i es n en e e re e i en n er i g n ips e re s e re s

r e e n e R r R represen i e r ng i ns n n e r ns r e in e er se en e i g n ips e er e r ng i ns ppe ring in s se en e n e R r R represen i e n eir r i n ers n e e se n

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An Extension of Cauchy's Arm Lemma with Application to Curve Development

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Abstract. Cauchy's "Arm Lemma" may be generalized to permit nonconvex "openings" of a planar convex chain. Although this (and further extensions) were known, no proofs have appeared in the literature. Here two induction proofs are offered. The extension can then be employed to establish that a curve that is the intersection of a plane with a convex polyhedron "develops" without self-intersection.

1 Introduction

Cauchy used his famous "Arm Lemma" to establish the rigidity of convex polyhedra [Cro97, p. 228]. Cauchy's lemma says that if $n-2$ consecutive angles of a convex polygon of n vertices are opened but not beyond π, keeping all but one edge length fixed and permitting that "missing" edge e to vary in length, then e lengthens (or retains its original length). This lemma has numerous applications, including at least one to curve development [OS89]. Here we describe an apparently little-known generalization of Cauchy's lemma to permit opening of the angles beyond π, as far reflex as they were originally convex. The conclusion remains the same: e cannot shorten. We then apply this extension to another result on curve development, that "slice curves" develop without self-intersection.

The first part of this paper (Section 2) concentrates on the generalization of Cauchy's lemma. The issue of self-intersection is addressed in Section 3, and the curve development result is discussed in Section 4.

2 Cauchy's Arm Lemma Extended

Let $A = (a_0, a_1, \ldots, a_n)$ be an n-link polygonal chain in the plane with n fixed edge lengths $\ell_i = |a_i a_{i+1}|$, $i = 0, \ldots, n-1$. We call the vertices a_i the *joints* of the chain, a_0 (which will always be placed at the origin) the *shoulder*, and a_n the *hand*. Define the *turn angle* α_i at joint a_i, $i = 1, \ldots, n-1$ to be the angle in $[-\pi, \pi]$ that turns the vector $a_i - a_{i-1}$ to $a_{i+1} - a_i$, positive for left (counterclockwise) and negative for right (clockwise) turns.

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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 280–291, 2001.

Define an open polygonal chain A to be *convex* if its joints determine a (nondegenerate) convex polygon, i.e., all joints are distinct points (in particular, $a_n \neq a_0$), all joints lie on the convex hull of A and they do not all lie on a line. Note there is no chain link between a_n and a_0. The turn angles for a convex chain all lie in $[0, \pi)$; but note this is not a sufficient condition for a chain to be convex, for it is also necessary that the angles at a_0 and a_n be convex.

We can view the configuration of a polygonal chain A to be determined by two vectors: the fixed edge lengths $L = (\ell_0, \ldots, \ell_{n-1})$ and the variable turn angles $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, with the convention that a_0 is placed at the origin and a_n horizontally left of a_0. Let $\mathcal{C}_L(\alpha) = A$ be the configuration so determined. We use α to represent the angles of the initial configuration, and β and γ to represent angles in a reconfiguration.

Let $D(A)$ be the open disk of radius $|a_n a_0|$ (the length of the "missing" link) centered on the shoulder joint a_0. We will call $D(A)$ the *forbidden (shoulder) disk*. We may state Cauchy's arm lemma in the following form:

Theorem 0. Cauchy's Arm Lemma. *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [0, \alpha_i] \tag{1}$$

we must have $|b_n b_0| \geq |a_n a_0|$, i.e., the hand cannot enter the forbidden disk $D(A)$.

Cauchy's lemma is sometimes known as Steinitz's lemma, because Steinitz noticed and corrected an error in the proof a century after Cauchy [Cro97, p. 235]. Many proofs of Cauchy's lemma are now known, e.g., [SZ67,Sin97] and [AZ98, p. 64].

Our generalization of Cauchy's lemma replaces the 0 in Eq. (1) by $-\alpha_i$, and is otherwise identical:

Theorem 1. Cauchy Extension. *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [-\alpha_i, \alpha_i] \tag{2}$$

we must have $|b_n b_0| \geq |a_n a_0|$, i.e., the hand cannot enter the forbidden disk $D(A)$.

The intuition, illustrated in Fig. 1, is perhaps best captured by Chern [Che67, p. 36]: "if an arc is 'stretched,' the distance between its endpoints becomes longer." See also our Java applet illustrating the theorem and its corollaries.[1] Although the chain may become nonconvex, Eq. (2) ensures that the movement constitutes a form of straightening. Note that Theorem 1 makes no claim about

[1] `http://cs.smith.edu/~orourke/Cauchy/` .

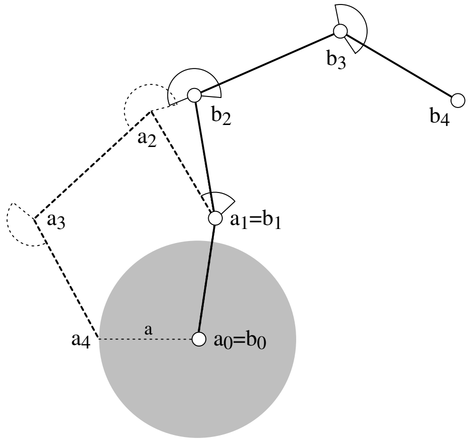

Fig. 1. A reconfiguration of a 4-link convex chain A to chain B, satisfying Eq. (2), leaves b_n outside the forbidden disk (shaded). The valid angle ranges are marked by circular arcs.

steadily increasing hand-shoulder separation during some continuous movement to B; indeed a continuous opening could first increase and later decrease the separation. Rather the claim is that a final configuration satisfying Eq. (2) cannot place the hand in the forbidden disk.

As with Cauchy's arm lemma, one may expect many different proofs of such a fundamental result. We sketch three proofs of Theorem 1 here; for full details, see [O'R00]. First we discuss a proof by S. S. Chern, followed by two induction proofs.

2.1 Chern's Proof of Theorem 1

There are three possible generalizations of Cauchy's Theorem 0: to smooth curves, to space curves, and to nonconvex angles. All three generalizations hold, and there is some confusion in the literature on when each was established. Certainly the primary source is acknowledged to be a 1921 paper by Axel Schur [Sch21]. Connelly mentions in [Con82, p. 30] that Schur generalized Cauchy's lemma to the smooth case. Some other mentions of Schur's theorem in the literature, e.g., in [Gug63, p.31], phrase it as the smooth, planar equivalent of Cauchy's lemma, which does not capture the nonconvexity permitted in the statement of Theorem 1. But others, notably the exposition by Chern [Che67], state it as a generalization also to space curves, and employ the absolute value of curvature, implicitly permitting nonconvexity.

Schur states his main result as follows:

> SATZ: "Verbiegt man eine ebene doppelpunktlose Kurve, die mit ihrer Sehne einen konvexen geschlossenen doppelpunktlosen Linienzug bildet, so wird dabei die Sehne länger."

> Roughly: "If one deforms a simple curve in the plane that (together with its chord) forms a convex simple closed curve, then the chord becomes longer."

Later he makes clear that his "deformation" is deformation into space without changing the curvature: "By deformation of a curve, we mean that after deforming the tangent planes the lengths and angles of the line segments should remain unchanged. If the curvature of the curve is continuous, then this is equal to demanding that the curvature doesn't change." It seems, then, that Schur did not generalize the opening aspect of Cauchy's lemma (to which he does not refer), but rather extended to deformations into space without angle changes. Moreover, it appears his proof (and claim) are for polygonal curves, not smooth curves.

Chern states Schur's theorem as follows [Che67, p. 36]:

> THEOREM: Let C be a plane arc with the curvature $k(s)$ which forms a convex curve with its chord AB. Let C^* be an arc of the same length referred to the same parameter s such that its curvature $k^*(s) \leq k(s)$. If d^* and d are the lengths of the chords joining their endpoints, then $d \leq d^*$.

He makes clear that by "curvature" he means the absolute value of the curvature, thus explicitly including nonconvex openings. Although he attributes the theorem to Schur, neither his theorem statement, and certainly not his proof, match Schur's. Let us then refer to this full generalization of Cauchy's theorem as the Chern-Schur theorem.

Chern's proof is carried out largely in the domain of the "tangent indicatrixes" of C and C^*, and relies on smoothness of the curves. He claims the theorem for "sectionally smooth curves" [p. 39], e.g., for polygonal curves, but without proof. Thus, as far as I can determine, no proof of Theorem 1 has appeared in the literature.

However, Chern's differential geometry proof may be specialized to our planar, nonsmooth instance, and although some of its elegance is lost, it can be carried through to establish Theorem 1 [O'R00]. Therefore Theorem 1 follows in spirit if not in fact from the Chern-Schur theorem.

The next section proves the theorem via an induction proof that makes explicit many relationships only implicit in Chern's proof.

2.2 First Induction Proof of Theorem 1

Although we impose no restriction on self-intersection of the chain, we will show in Theorem 3 that the chain remains simple. Note that, because we fix a_0 to the origin, and the first turn angle is at joint a_1, in any reconfiguration the first edge of the chain is fixed.

The first induction proof of Theorem 1 requires a few preparatory lemmas, whose proofs may be found in [O'R00]. We start with the simple observation that negating the turn angles reflects the chain.

Lemma 1. *If a chain $A = \mathcal{C}_L(\alpha)$ is reconfigured to $B = \mathcal{C}_L(\beta)$ with $\beta_i = -\alpha_i$, $i = 1, \ldots, n-1$, then B is a reflection of A through the line M containing $a_0 a_1$, and $|b_n b_0| = |a_n a_0|$.* □

Call a reconfiguration $B = \mathcal{C}_L(\beta)$ of a convex chain $A = \mathcal{C}_L(\alpha)$ which satisfies the constraints of Eq. (2) a *valid reconfiguration*, and call the vector of angles β *valid angles*. Define the *reachable region* $R_L(\alpha)$ for a convex chain $A = \mathcal{C}_L(\alpha)$ to be the set of all hand positions b_n for any valid reconfiguration $B = \mathcal{C}_L(\beta)$. One can view Theorem 1 as the claim that $R_L(\alpha) \cap D(A) = \emptyset$. (Again, see our applet[1] for an illustration of this claim.) It is well known [HJW84][O'R98, p. 326] that the reachable region for a chain with no angle constraints is a shoulder-centered closed annulus, but angle-constrained reachable regions seem unstudied.

For the first proof we need two technical lemmas.

Lemma 2. *The configuration of a chain $A = \mathcal{C}_L(\alpha)$ is a continuous function of its turn angles α.* □

Lemma 3. *$R_L(\alpha)$ is a closed set.* □

We use this lemma to help identify, among potential counterexamples, the "worst" violators. Define a configuration $B = \mathcal{C}_L(\beta)$ to be *locally minimal* if there is a neighborhood N of β such that, for all $\beta' \in N$, the determined hand position b'_n is no closer to the shoulder: $|b'_n a_0| \geq |b_n a_0|$. Thus the hand's distance to the shoulder is locally minimal.

Lemma 4. *Let $B = \mathcal{C}_L(\beta)$ be a reconfiguration of convex chain $A = \mathcal{C}_L(\alpha)$ with $b_n \in D(A)$. Then either $b_n = a_0$, or there is some locally minimal configuration $B' = \mathcal{C}_L(\beta')$ with $b'_n \in D(A)$.* □

The above lemma will provide a "hook" to reduce n in the induction step. We separate out the base of the induction in the next lemma.

Lemma 5. *Theorem 1 (Cauchy Extension) holds for $n = 2$.*
Proof: A 2-link chain's configuration is determined by single angle at a_1. The reachable region is a single circular arc exterior to $D(A)$, centered on a_1, of radius ℓ_1. See Fig. 2. □
We now prove Theorem 1 by induction.
Proof: Lemma 5 establishes the theorem for $n = 2$. Assume then that the theorem holds for all chains of $n-1$ or fewer links. We seek to establish it for an n-link chain $A = \mathcal{C}_L(\alpha)$, $n > 2$. Assume, for the purposes of contradiction, that A may be reconfigured so that the hand falls inside the forbidden disk $D(A)$. We seek a contradiction on a shorter chain. By Lemma 4, one of two cases holds: the hand reaches a_0, or there is a locally minimal configuration.

1. Suppose $B = \mathcal{C}_L(\beta)$ is such that $b_n = a_0$, as illustrated in Fig. 3(b). We know that $\ell_{n-1} = |a_{n-1} a_n| < |a_{n-1} a_0|$ by the triangle inequality: By definition, A forms a nondegenerate convex polygon, so the triangle $\triangle(a_0, a_{n-1}, a_n)$ is

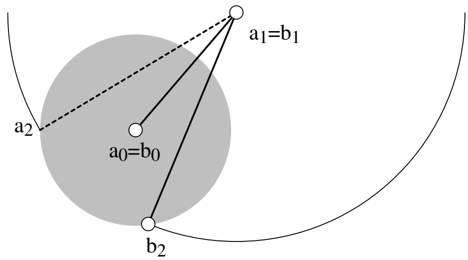

Fig. 2. $R_L(\alpha)$ for a 2-link chain is a circle arc centered on $a_1 = b_1$.

nondegenerate (see Fig. 3(a)). Now consider the chains A' and B' obtained by removing the last links $a_{n-1}a_n$ and $b_{n-1}b_n$. First, A' is a convex chain of $n-1$ links, so the induction hypothesis applies and says that A' cannot be validly reconfigured to place b_{n-1} closer to a_0 than $a' = |a_{n-1}a_0|$. B' places b_{n-1} at distance ℓ_{n-1} from a_0, which we just observed is less than a'. It remains to argue that B' is a valid reconfiguration of A', i.e., that it satisfies Eq. (2). However, this is satisfied for $i = 1, \ldots, n-2$ because these angles are not changed by the shortening, and after shortening there is no constraint on β_{n-1}. Thus B' is a valid reconfiguration of A' but places the hand in the forbidden disk $D(A')$, a contradiction.

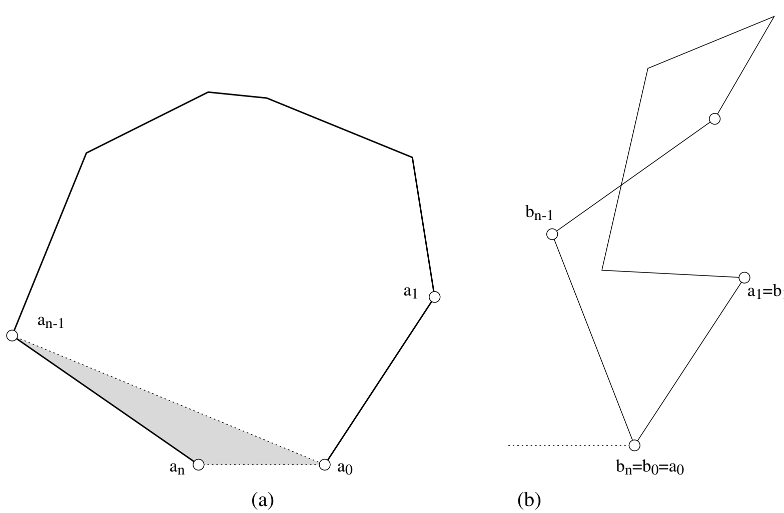

Fig. 3. Case 1: $b_n = a_0$. (Drawing (b) is not accurate.)

2. We may henceforth assume, by Lemma 4, that there is a locally minimal configuration $B = \mathcal{C}_L(\beta)$ that places $b_n \in D(A)$. Again we seek to shorten the chain and obtain a contradiction.

First we establish that at least one[2] β_i is at the limit of its valid turn range: $\beta_i = \pm\alpha_i$. Suppose to the contrary that all β_i, $i = 1, \ldots, n-1$, are strictly interior to their allowable turn ranges: $\beta_i \in (-\alpha_i, \alpha_i)$. Let M be the line containing $b_0 b_n$. Consider two cases:

(a) Some b_i, $i = 1, \ldots, n-1$, does not lie on M. Then because β_i is not extreme, the subchain $(b_{i+1}, \ldots, b_n)$ may be rotated about b_i in both directions. Because b_i is off M, one direction or the other must bring b_n closer to b_0, contradicting the fact that b_n is locally minimal.

(b) All b_i lie on M. Then there must be some b_i which is extreme on M. For this b_i, $\beta_i = \pm\pi$. But $\alpha_i \in [0, \pi)$: the nondegeneracy assumption bounds α_i away from π, and so bounds β_i away from $\pm\pi$.

Henceforth let b_i be a joint whose angle β_i is extreme. If $\beta_i = -\alpha_i$, then reflect B about $b_0 b_n$ so that $\beta_i = \alpha_i$ is convex. By Lemma 1, this does not change the distance from b_n to the shoulder, so we still have $b_n \in D(A)$.

We are now prepared to shorten the chains. Let A' and B' be the chains resulting from removing a_i and b_i from A and B respectively:

$$A' = (a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \tag{3}$$
$$B' = (b_0, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) \tag{4}$$

A crucial point to notice is that $|b_{i-1} b_{i+1}| = |a_{i-1} a_{i+1}|$ because $\beta_i = \alpha_i$; this was the reason for focusing on an extreme β_i. Therefore B' is a reconfiguration of A'. Of course both A' and B' contain $n-1$ links, so the induction hypothesis applies. Moreover, because $i \leq n-1$, the b_i removed does not affect the position of b_n. So $b_n \in D(A)$ by hypothesis. To derive a contradiction, it only remains to show that B' is a valid reconfiguration of A', i.e., one that satisfies the turn constraints (2).

Let α'_{i+1} be the turn angle at a_{i+1} in A'. We analyze this turn angle in detail, and argue later that the situation is analogous at a_{i-1}. Let θ be the angle of the triangle $\triangle_i = \triangle(a_i, a_{i+1}, a_{i-1})$ at a_{i+1}; see Fig. 4(a). Because A is a convex chain, cutting off $\triangle_i$ from A increases the turn angle at a_{i+1} in A':

$$\alpha'_{i+1} = \theta + \alpha_{i+1} \tag{5}$$

Now consider the turn angle β'_{i+1} at b_{i+1} in B'. Although here the turn could be negative, as in Fig. 4(b), it is still the case that the turn is advanced by θ by the removal of $\triangle_i$:

$$\beta'_{i+1} = \theta + \beta_{i+1} \tag{6}$$

We seek to prove that $\beta'_{i+1} \in [-\alpha'_{i+1}, \alpha'_{i+1}]$. Substituting the expressions from Eqs. (5) and (6) into the desired inequality yields:

$$-\alpha_{i+1} - 2\theta \leq \beta_{i+1} \leq \alpha_{i+1} \tag{7}$$

And this holds because $\theta > 0$ and $\beta_{i+1} \in [-\alpha_{i+1}, \alpha_{i+1}]$ (because B is a valid reconfiguration of A). The intuition here is that the nesting of the turn angle

[2] In fact I believe that all must be extreme, but the proof only needs one.

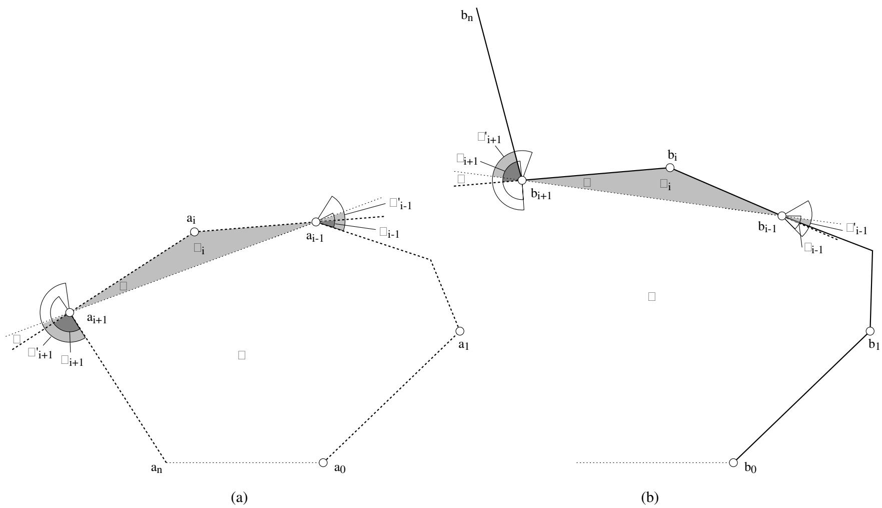

Fig. 4. (a) Shortening the chain A by removal of a_i determines new, larger turn angles α'_{i+1} and α'_{i-1} at a_{i+1} and a_{i-1} respectively. (b) Here the turn angles β_{i+1} and β'_{i+1} are negative.

ranges at a_{i+1} in A and A' (evident in Fig. 4(a)) carries over, rigidly attached to $\triangle_i$, to B, so that satisfying the tighter constraint in B also satisfies the looser constraint in B'.

Although the situation is superficially different at a_{i-1} because our definition of turn angle depends on the orientation of the chain, it is easily seen that the turn constraint is identical if the orientation is reversed.

We have thus established that B' is a valid reconfiguration of A'. By the induction hypothesis, the hand b_n of B' cannot enter the forbidden disk $D(A')$. By assumption $b_n \in D(A)$. But note that $D(A') = D(A)$, because, as mentioned above, deleting a_i and b_i does not affect the positions of a_n and b_n respectively. This contradiction shows that our assumption that $b_n \in D(A)$ cannot hold, and establishes the theorem.

□

The following corollary extends the distance inequality to every point of the chain.

Corollary 2 *Let $A = \mathcal{C}_L(\alpha)$ be a convex chain as in Theorem 1, and let p_1, $p_2 \in A$ be any two distinct points of the chain. Then in any valid reconfiguration B, the points q_1, $q_2 \in B$ corresponding to p_1 and p_2 satisfy $|q_1q_2| \geq |p_1p_2|$, i.e., they have not moved closer to one another.* □

2.3 Second Induction Proof of Theorem 1

We now sketch a second proof, which avoids reliance on locally minimal configurations. Unlike the previous proof, this one employs Cauchy's Arm Lemma

(Theorem 0), rather than proving it along the way. The proof is again inductive, by contradiction from a shortened chain, and relies on the same detailed argument concerning the turn angle ranges. None of those details will be repeated.

Proof: Let $A = \mathcal{C}_L(\alpha)$ be the given convex chain, and $C = \mathcal{C}_L(\gamma)$ a valid reconfiguration that places $c_n \in D(A)$, in contradiction to the theorem. We first construct an "intermediate" configuration $B = \mathcal{C}_L(\beta)$ with $\beta_i = |\gamma_i|$ for all $i = 1, \ldots, n-1$, i.e., B is a convex chain formed by flipping all turns in C to be positive. Note that, because γ is a valid angle vector for A, $\gamma_i \in [-\alpha_i, \alpha_i]$, and so $\beta_i \in [0, \alpha_i]$. As this is exactly the Cauchy arm opening condition, Eq. (1), we may apply Theorem 0 to conclude that $b = |b_n b_0| \geq |a_n a_0| = a$.

Now we focus attention on chains B and C. Because $\gamma_i = \pm\beta_i$, $\gamma_i \in [-\beta_i, \beta_i]$. Therefore, C is a valid reconfiguration of B. But here is the point: every angle γ_i of C is extreme with respect to B, and so there is no need to invoke local minimality.

Choose an i and remove b_i from B and c_i from C, obtaining shorter chains B' and C'. Applying the argument from the previous section verbatim, we conclude that C' is a valid reconfiguration of B'. But because B' has $n-1$ links, the induction hypothesis applies and shows that c_n cannot enter the forbidden disk $D(b)$, with $b = |b_n b_0|$. Because $b \geq a$, c_n cannot be in $D(A)$ either. This contradicts the assumption and establishes the theorem. □

3 Noncrossing of Straightened Curve

Define a polygonal chain to be *simple* if nonadjacent segments are disjoint, and adjacent segments intersect only at their single, shared endpoint. By our nondegeneracy requirement, convex chains are simple. In particular, any opening of a convex chain via Cauchy's arm lemma (Theorem 0) remains simple because it remains convex. We now establish a parallel result for the generalized straightening of Theorem 1. We generalize slightly to permit the convex chain to start with the hand at the shoulder.

Theorem 3. *If $A = (a_0, \ldots, a_n) = \mathcal{C}_L(\alpha)$ is a closed convex chain with n fixed edge lengths L and turn angles α, closed in the sense that $a_n = a_0$, then any valid reconfiguration to $B = \mathcal{C}_L(\beta)$ is a simple polygonal chain.*

Proof: Suppose to the contrary that B is nonsimple. Let q_2 be the first point of B, measured by distance along the chain from the shoulder b_0, that coincides with an earlier point $q_1 \in B$. Thus q_1 and q_2 represent the same point of the plane, but different points along B. See Fig. 5. Because B is nonsimple, these "first touching points" exist, and we do not have both $q_1 = b_0$ and $q_2 = b_n$ (because that would make B a simple, closed chain). Let p_1 and p_2 be the points of A corresponding to q_1 and q_2.

Corollary 2 guarantees that $|q_1 q_2| \geq |p_1 p_2|$. But $|q_1 q_2| = 0$, and because the q's do not coincide with the original hand and shoulder, $|p_1 p_2| > 0$. This contradiction establishes the claim. □

One could alternatively prove this theorem by induction on the length of the chain, showing that in a continuous motion to B, the first violation of simplicity is either impossible by the induction hypothesis, or directly contradicts Theorem 1.

Corollary 4 *A valid reconfiguration of an open convex chain remains simple.*
Proof: The proof of Theorem 3 guarantees that even the final missing edge is not crossed, so the corollary is obtained by simply ignoring that last edge. □

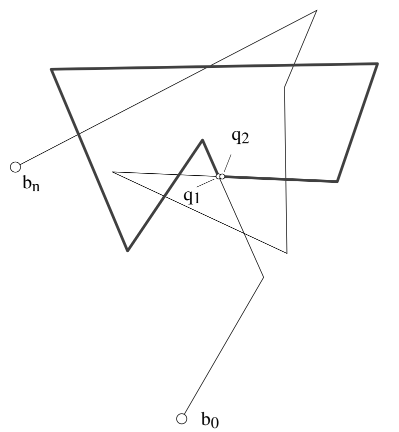

Fig. 5. Violation of Theorem 1. $q_1 = q_2$ is the first point of self-contact; the initial portion of B, up to q_2, is highlighted.

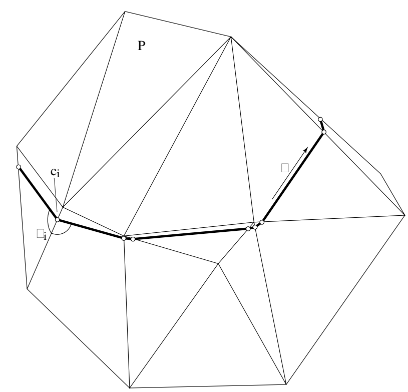

Fig. 6. Γ is the intersection of a plane (not shown) with polyhedron P.

4 Application to Curve Development

Although the intersection of a plane Π with a convex polyhedron P is a convex polygon Q within that plane, on the surface of P, this *slice curve* (Fig. 6) can be nonconvex, turning both left and right. The "development" of a curve on a plane is determined by its turning behavior on the surface. Thus slice curves develop (in general) to nonconvex, open chains on a plane. An earlier result is that a closed convex polygonal curve on a convex polyhedron, i.e., one whose turns are *all* leftward on the surface, develops to a simple polygonal chain [OS89]. That proof relied on Cauchy's Arm Lemma (Theorem 0). We claim that slice curves also develop without self-intersection, despite their nonconvexity. The proof relies on Theorem 1, the extension of Cauchy's lemma.

Let Γ be an oriented slice curve on the surface of P. Let $c_0, c_1, \ldots, c_n$ be the *corners* of Γ, the points at which Γ crosses a polyhedron edge with a dihedral angle different from π, or meets a polyhedron vertex. Define the *right surface angle* $\theta(p)$ at a point $p \in \Gamma$ to be the total incident face angle at p to the right of the directed curve Γ at p. Only at a corner c_i of Γ is the right surface angle θ_i different from π. Note that θ_i could be greater or less than π, i.e., the slice curve could turn right or left on the surface.

Define the *right development* of Γ to be a planar drawing of the polygonal chain Γ as the chain $B = (b_0, b_1, \ldots, b_n)$ with the same link lengths, $|b_i b_{i+1}| = |c_i c_{i+1}|$ for $i = 0, \ldots, n-1$, and with exterior angle θ_i to the right of b_i the same as the surface angle to the right of Γ at c_i on P, for all $i = 1, \ldots, n-1$. Left development is defined similarly, with any curve "between" considered *a* development of Γ. If Γ avoids all polyhedron vertices, then the left and right developments are identical, and so the development of Γ is unique.

In [O'R01] I prove that the developed chain B is a valid reconfiguration (in the sense used in Section 2.2, i.e., satisfies Eq. (2)) of the chain A representing Q in Π. Via Corollary 4, this leads to:

Theorem 5. *A slice curve $\Gamma = P \cap \Pi$, the intersection of a convex polyhedron P with a plane Π, develops on a plane to a simple (noncrossing) polygonal curve.*

Because the Chern-Schur Theorem encompasses smooth curves, Theorem 5 should generalize to slice curves for any convex body B.

Acknowledgments

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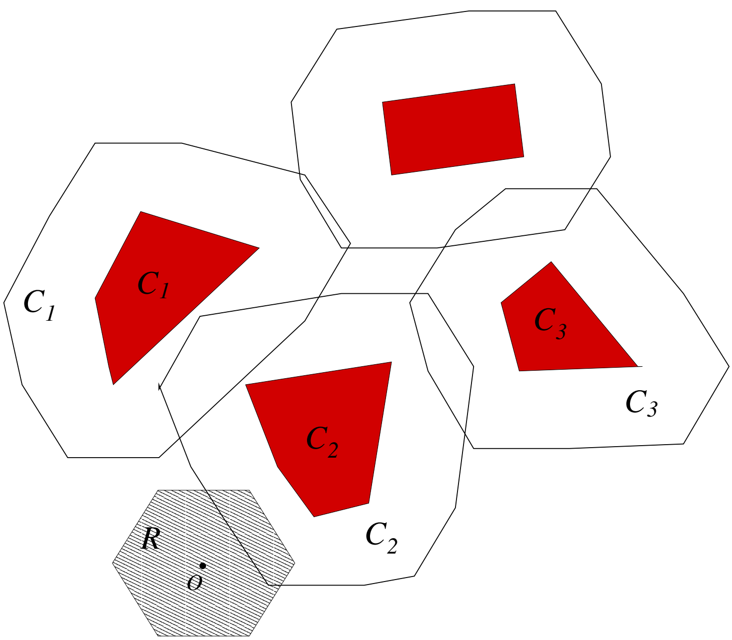

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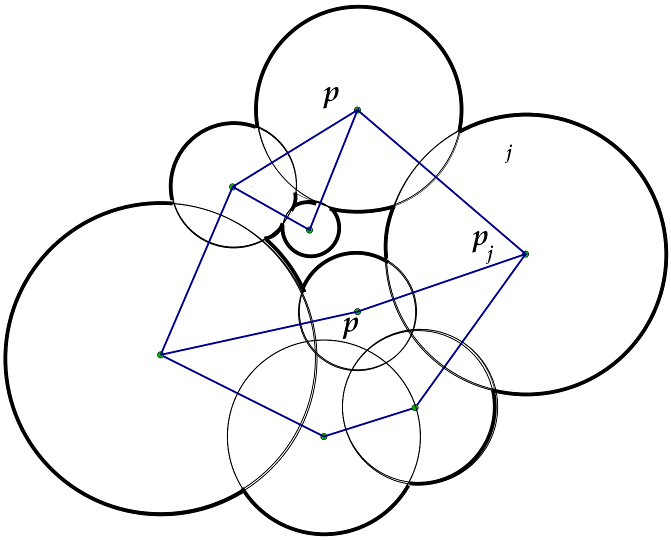

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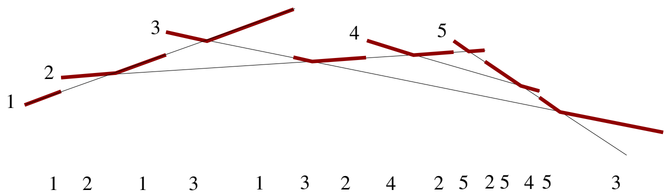

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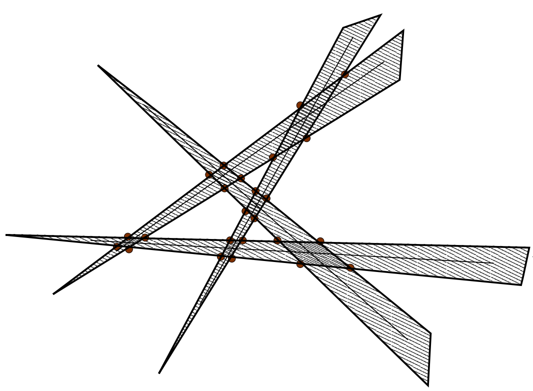

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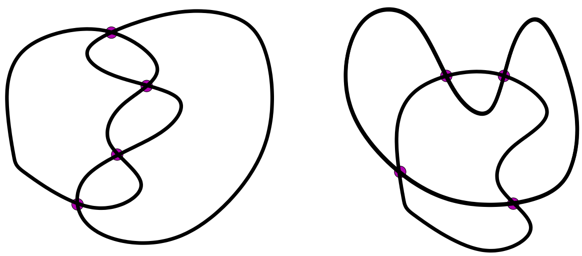

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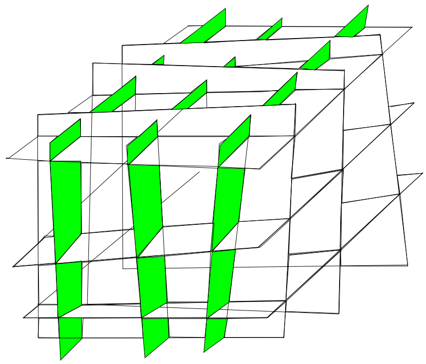

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Structure Theorems for Systems of Segments

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Abstract. We study intersection properties of systems of segments in the plane. In particular, we show that there exists a constant $c > 0$ such that every system $\mathcal{S}$ of n straight-line segments in the plane has two at least cn-element subsystems $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that either every segment in $\mathcal{S}_1$ intersects all elements of $\mathcal{S}_2$, or no segment in $\mathcal{S}_1$ intersects any element of $\mathcal{S}_2$. We also propose a fast approximate solution for reporting *most* intersections among n segments in the plane.

1 Introduction

The problem of detecting and reporting intersections among straight-line segments in the plane is one of the oldest and most extensively studied topics in computational geometry. It is a basic ingredient of many hidden surface removal algorithms, and has numerous other applications in computer graphics, motion planning, geographic information systems, etc. The first efficient techniques were developed by Shamos and Hoey [SH76] and Bentley and Ottmann [BO79] more than twenty years ago. The running times of the best known algorithms, due to Balaban [B95] and Chazelle and Edelsbrunner [CE92], are $O(n \log n + I)$, where n and I denote the number of segments and the number of intersections, resp. (See also [PS91].)

In the present paper, we discuss some structural properties of *intersection graphs* of segments, i.e., graphs that can be obtained by assigning a vertex to every element of a system of segments $\mathcal{S}$ in the plane, and connecting two of them by an edge if and only if their intersection is non-empty. Throughout this paper, we assume that the elements of $\mathcal{S}$ are in *general position*, i.e., no two segments are parallel and no three of their endpoints are collinear. In particular, if two elements of $\mathcal{S}$ intersect, then they determine a proper crossing.

We prove the following Ramsey-type result.

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Theorem 1. *There exists a constant $C > 0$ such that every system $\mathcal{S}$ of n segments in the plane has two disjoint subsystems $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq Cn$ and*

(i) either every segment in $\mathcal{S}_1$ crosses all segments in $\mathcal{S}_2$,

(ii) or no segment in $\mathcal{S}_1$ crosses any segment in $\mathcal{S}_2$.

In the sequel, A stands for an absolute constant smaller than 10^6. Theorem 1 is a direct corollary of the following two complementary results.

Theorem 2. *Any system $\mathcal{S}$ of n segments in the plane with at least cn^2 crossings ($c > 0$) has two disjoint subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq \frac{(2c)^A}{660} n$ and every segment in $\mathcal{S}_1$ crosses all segments in $\mathcal{S}_2$.*

Theorem 3. *Any system $\mathcal{S}$ of n segments in the plane with at least cn^2 non-crossing pairs ($c > 0$) has two disjoint subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq \frac{(c/5)^A}{330} n$ and no segment in $\mathcal{S}_1$ crosses any segment in $\mathcal{S}_2$.*

The above results, combined with Szemerédi's Regularity Lemma [S78], can be used to establish a fairly strong structure theorem for intersection graphs of segments. We say that two sets have *almost the same* number of elements if their sizes differ by at most a factor of 2.

Theorem 4. *For any $\varepsilon > 0$, there exists an integer $K = K(\varepsilon)$ with the property that any system $\mathcal{S}$ of segments in the plane can be partitioned into $K + 1$ subfamilies, $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_K$ such that $|\mathcal{S}_0| < \varepsilon|\mathcal{S}|$, all other subfamilies have almost the same size, and for all but at most εK^2 pairs $1 \leq i, j \leq K$,*

(i) either every segment in $\mathcal{S}_i$ crosses all segments in $\mathcal{S}_j$,

(ii) or no segment in $\mathcal{S}_i$ crosses any segment in $\mathcal{S}_j$.

Fix an element s_i in each $\mathcal{S}_i$. For any $s \in \mathcal{S}$, let $f(s) := s_i$ if and only if s belongs $\mathcal{S}_i$ ($0 \leq i \leq K$). We can think of $f(s)$ as the segment *representing* s. According to Theorem 4, with a very small error, two randomly selected elements $s, t \in \mathcal{S}$ cross each other if and only if $f(s) \cap f(t)$ is non-empty.

A *geometric graph* is a graph whose vertices are points in general position in the plane (i.e., no three points are on a line) and whose edges are straight-line segments connecting these points. Our last two results are easy corollaries to Theorems 2 and 3, respectively.

Theorem 5. *Any geometric graph G with n vertices and at least cn^2 edges ($c > 0$) has two disjoint sets of edges $E_1, E_2 \subset E(G)$ such that $|E_1|, |E_2| \geq (c/32)^{A+3}\binom{n}{2}$ and every edge in E_1 crosses all edges in E_2.*

Theorem 6. *Any geometric graph G with n vertices and at least cn^2 edges ($c > 0$) has two disjoint sets of edges $E_1, E_2 \subset E(G)$ such that $|E_1|, |E_2| \geq (c/34)^{A+3}\binom{n}{2}$ and no edge in E_1 crosses any edge in E_2.*

The rest of the paper is organized as follows. In Section 2, we establish Theorems 2 and 3. Theorem 4 is proved in Section 3. The last section contains the proofs of Theorems 5 and 6, as well as some concluding remarks.

2 Proofs of Theorems 2 and 3

Three sets of points in the plane are said to be *separable* if each of them can be separated from the other two by a straight line. Given three separable sets, there is no straight line which intersects the convex hull of all of them.

Lemma 1. *Every set of n points in general position in the plane has three separable subsets of size $\lfloor n/6 \rfloor$.*

Proof. Assume without loss of generality that n is divisible by 6, and let P be an n-element point set. Choose two lines that divide the plane into 4 regions, containing $n, 2n, n$, and $2n$ points of P in their interiors, in this cyclic order. Let P_1, P_2, P_3, and P_4 denote the corresponding subsets of P. By the *ham-sandwich theorem*, there is a line ℓ which simultaneously cuts P_2 and P_4 into two halves of equal size (see Fig. 1). Then ℓ avoids either the convex hull of P_1 or that of P_3. Assume, by symmetry, that P_1 is 'above' ℓ. Then P_1 and the parts of P_2 and P_4 'below' ℓ are three separable sets. □

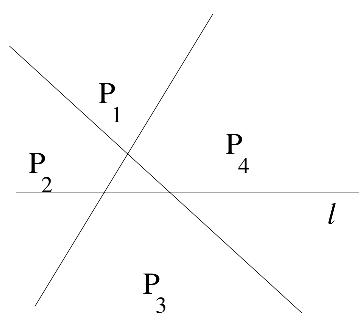

Fig. 1.

Lemma 2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two systems of segments in general position in the plane. Then there are two subsystems $\mathcal{S}^* \subseteq \mathcal{S}$, $\mathcal{T}^* \subseteq \mathcal{T}$ such that $|\mathcal{S}^*| \geq \lfloor |\mathcal{S}|/330 \rfloor$, $|\mathcal{T}^*| \geq \lfloor |\mathcal{T}|/330 \rfloor$, and*

(i) either every segment in $\mathcal{S}^*$ crosses all segments in $\mathcal{T}^*$,

(ii) or no segment in $\mathcal{S}^*$ crosses any segment in $\mathcal{T}^*$.

Proof. Let $|\mathcal{S}| = m, |\mathcal{T}| = n$, and suppose, for simplicity, that both m and n are multiples of 330. Let P be the set of endpoints of all segments in $\mathcal{S}$. By Lemma 1, there are three separable $m/3$-element subsets, $P_1, P_2, P_3 \subseteq P$. Color a segment $t \in \mathcal{T}$ with color i if its supporting line does not intersect the convex hull of P_i $(i = 1, 2, 3)$. Let $\mathcal{T}_i$ denote the segments of color i. At least one third of the elements of $\mathcal{T}$ get the same color, so we can assume with no loss of generality that $|\mathcal{T}_1| \geq n/3$.

If there are at least $m/330$ segments in $\mathcal{S}$, both of whose endpoints belong to P_1, then we are done, because these segments are disjoint from all elements of $\mathcal{T}_1$.

Hence, we can assume that at least $(1/3 - 2/330)m = 18m/55$ elements of $\mathcal{S}$ have precisely one of their endpoints in P_1. Let Q denote the set of other endpoints of these segments. Let us choose three separable subsets $Q_1, Q_2, Q_3 \subseteq Q$, each of size at least $|Q|/6 = 3m/55$. Just as before, color a segment $t \in \mathcal{T}_1$ with color i if its supporting line does not intersect the convex hull of Q_i $(i = 1, 2, 3)$. Again, at least $|\mathcal{T}_1|/3 \geq n/9$ elements of $\mathcal{T}_1$ get the same color, say color 1; they form a subsystem $\mathcal{T}_{11} \subseteq \mathcal{T}_1$.

Let $\mathcal{S}_{11}$ denote set of all elements of $\mathcal{S}$ with one endpoint in P_1 and the other in Q_1. Clearly, we have $|\mathcal{S}_{11}| = |Q_1| \geq 3m/55$.

Let us repeat now the whole procedure with $\mathcal{T}_{11}$ in the place of $\mathcal{S}$ and $\mathcal{S}_{11}$ in the place of $\mathcal{T}$. We obtain two subsets, $\mathcal{T}' \subseteq \mathcal{T}_{11}$ and $\mathcal{S}' \subseteq \mathcal{S}_{11}$, satisfying

$$|\mathcal{T}'| \geq \frac{3|\mathcal{T}_{11}|}{55} \geq \frac{n}{165}, \quad |\mathcal{S}'| \geq \frac{|\mathcal{S}_{11}|}{9} \geq \frac{m}{165}.$$

We can assume that at least half of the supporting lines of the elements of $\mathcal{T}'$ cross the convex hull of $\mathcal{S}'$, for otherwise we would obtain two non-crossing systems of at least $|\mathcal{T}'|/2$ and $|\mathcal{S}'|$ segments. The set of all elements of $\mathcal{T}'$, whose supporting lines cross the convex hull of $\mathcal{S}'$ is denoted by $\mathcal{T}^*$. Similarly, we can assume that the supporting lines of at least half of the elements of $\mathcal{S}'$ cross the convex hull of $\mathcal{T}^*$; otherwise, we could find two non-crossing systems of at least $|\mathcal{T}^*|$ and $|\mathcal{S}'|/2$ segments. Let $\mathcal{S}^*$ denote the set of all elements of $\mathcal{S}'$, whose supporting lines cross the convex hull of $\mathcal{T}^*$. It follows from the definitions that every element of $\mathcal{S}^*$ crosses all elements of $\mathcal{T}^*$ and that

$$|\mathcal{S}^*| \geq \frac{|\mathcal{S}|'}{2} \geq \frac{m}{330}, \quad |\mathcal{T}^*| \geq \frac{|\mathcal{T}|'}{2} \geq \frac{n}{330}. \quad \square$$

Given any system of segments, $\mathcal{S}$ and $\mathcal{T}$, in general position in the plane, define their *crossing density*, $\delta(\mathcal{S}, \mathcal{T})$, as the number of crossing pairs (s, t), $s \in \mathcal{S}$, $t \in \mathcal{T}$ divided by $|\mathcal{S}| \cdot |\mathcal{T}|$. Clearly, we have $0 \leq \delta(\mathcal{S}, \mathcal{T}) \leq 1$.

Theorems 2 and 3 readily follow from the next result.

Theorem 7. *There exists a constant $A < 10^6$ satisfying the following condition. Let $\mathcal{S}$ and $\mathcal{T}$ be any sets of segments in general position in the plane, and suppose that their crossing density is at least $c > 0$. Then there are two disjoint subsystems $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{T}' \subseteq \mathcal{T}$ such that*

$$|\mathcal{S}'| \geq \frac{c^A}{330}|\mathcal{S}|, \quad |\mathcal{T}'| \geq \frac{c^A}{330}|\mathcal{T}|,$$

and every segment in $\mathcal{S}'$ crosses all segments in $\mathcal{T}'$.

Proof. Let $|\mathcal{S}| = m, |\mathcal{T}| = n$, and suppose first that both m and n are powers of 330. According to our assumption, $\delta(\mathcal{S}, \mathcal{T}) \geq c$.

Applying Lemma 2, we obtain two subsystems, $\mathcal{S}^* \subset \mathcal{S}$, $\mathcal{T}^* \subset \mathcal{T}$, such that $|\mathcal{S}^*| = m/330, |\mathcal{T}^*| = n/330$, and $\delta(\mathcal{S}^*, \mathcal{T}^*)$ is either 1 or 0. In the first case we are done, so assume $\delta(\mathcal{S}^*, \mathcal{T}^*) = 0$. Then we have

$$c \leq \delta(\mathcal{S}, \mathcal{T}) = \frac{329}{330^2}\delta(\mathcal{S}, \mathcal{T} - \mathcal{T}^*) + \frac{329}{330^2}\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T}) + \frac{329^2}{330^2}\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T} - \mathcal{T}^*).$$

Therefore, at least one of the crossing densities $\delta(\mathcal{S}, \mathcal{T} - \mathcal{T}^*)$, $\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T})$, $\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T} - \mathcal{T}^*)$ exceeds

$$c_1 := c\frac{330^2}{330^2 - 1}.$$

In other words, there exist two subsystems, $\mathcal{S}_1 \subset \mathcal{S}$, $\mathcal{T}_1 \subset \mathcal{T}$, with $|\mathcal{S}_1| \geq m/330$, $|\mathcal{T}_1| \geq n/330$ such that $\delta(\mathcal{S}_1, \mathcal{T}_1) \geq c_1$.

Applying Lemma 2 to $\mathcal{S}_1$ and $\mathcal{T}_1$, we obtain two subsystems $\mathcal{S}^{**} \subset \mathcal{S}_1$, $\mathcal{T}^{**} \subset \mathcal{T}_1$, such that $|\mathcal{S}^{**}| \geq m/330^2, |\mathcal{T}^{**}| \geq n/330^2$, and $\delta(\mathcal{S}^{**}, \mathcal{T}^{**})$ is either 1 or 0. Again, we can assume that $\delta(\mathcal{S}^{**}, \mathcal{T}^{**}) = 0$, otherwise we are done. As before, we can find two subsystems, $\mathcal{S}_2 \subset \mathcal{S}_1$, $\mathcal{T}_2 \subset \mathcal{T}_1$, with $|\mathcal{S}_2| \geq m/330^2$, $|\mathcal{T}_2| \geq n/330^2$ such that

$$\delta(\mathcal{S}_2, \mathcal{T}_2) \geq c_2 := c\left(\frac{330^2}{330^2 - 1}\right)^2.$$

Since the crossing density between any two sets is at most 1, after some

$$k \leq \frac{\log \frac{1}{c}}{\log \frac{330^2}{330^2 - 1}}$$

steps, this procedure will terminate. That is, when we apply Lemma 2 for the k-th time, we obtain two subsystems $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{T}' \subseteq \mathcal{T}$ such that $|\mathcal{S}'| \geq m/330^k$, $|\mathcal{T}'| \geq n/330^k$, and $\delta(\mathcal{S}', \mathcal{T}') = 1$. Thus, every element of $\mathcal{S}'$ crosses all elements of $\mathcal{T}'$, and $|\mathcal{S}'| \geq c^A m$, $|\mathcal{T}'| \geq c^A n$, where

$$A \leq \frac{\log 330}{\log \frac{330^2}{330^2 - 1}} < 10^6.$$

This completes the proof of Theorem 7 in the case when m and n are powers of 330. Otherwise, using an easy averaging argument, we can find $\mathcal{S}_0 \subseteq \mathcal{S}$, $\mathcal{T}_0 \subseteq \mathcal{T}$, whose sizes are powers of 330, $|\mathcal{S}_0| \geq m/330$, $|\mathcal{T}_0| \geq n/330$, and $\delta(\mathcal{S}_0, \mathcal{T}_0) \geq c$. Applying the above argument to $\mathcal{S}_0$ and $\mathcal{T}_0$, the result follows. □

Proof of Theorem 2. Assume, for simplicity, that n is even. Given a system of n segments in general position in the plane, which determine at least cn^2 crossings, one can partition it into two equal parts so that the crossing density between them is at least $2c$ (see e.g. [PA95]). Applying Theorem 7 to these parts, the result follows. □

Theorem 3 can be established analogously, by repeated application of Lemma 2. However, here we deduce it from Theorems 2 and 3.

Proof of Theorem 3. Let $\mathcal{S}$ be a set of n segments in general position in the plane with at least cn^2 non-crossing pairs. For any $s \in \mathcal{S}$, let $\ell(s)$ denote the supporting line of s. The set $\ell(s) \setminus s$ consists of two half-lines; denote them by $h_1(s)$ and $h_2(s)$. Let $\mathcal{H}_1 := \{h_1(s) : s \in \mathcal{S}\}$, $\mathcal{H}_2 := \{h_2(s) : s \in \mathcal{S}\}$, $\mathcal{T} := \mathcal{S} \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Further, for any $h \in \mathcal{H}_1 \cup \mathcal{H}_2$, let $s(h)$ be the unique segment $s \in \mathcal{S}$, for which $h_1(s)$ or $h_2(s)$ is equal to s.

Note that if two segments $s, t \in \mathcal{S}$ do not cross each other, then the crossing between their supporting lines, $\ell(s)$ and $\ell(t)$, gives rise to a crossing between a pair of elements of $\mathcal{T}$, involving at least one half-line. Therefore, the number of crossing pairs in $\mathcal{T}$ involving at least one half-line is at least cn^2. There are three possibilities:

1. for some $i = 1, 2$, the number of crossing pairs in $\mathcal{H}_i$ is at least $cn^2/5$;
2. the number of crossing pairs between $\mathcal{H}_1$ and $\mathcal{H}_2$ is at least $cn^2/5$;
3. for some $i = 1, 2$, the number of crossing pairs between $\mathcal{H}_i$ and $\mathcal{S}$ is at least $cn^2/5$.

In Case 1, applying Theorem 2 to $\mathcal{H}_i$, we obtain two subsystems, $\mathcal{H}_{i1}, \mathcal{H}_{i2} \subset \mathcal{H}$, whose sizes are at least $\frac{(2c/5)^A}{660} n > \frac{(c/5)^A}{330}$, and every half-line in $\mathcal{H}_{i1}$ crosses all half-lines in $\mathcal{H}_{i2}$. Then $\mathcal{S}_1 := \{s(h) : h \in \mathcal{H}_{i1}\}$ and $\mathcal{S}_2 := \{s(h) : h \in \mathcal{H}_{i2}\}$ meet the requirements in Theorem 3.

In Case 2, apply Theorem 7 to obtain $\mathcal{H}'_1 \subseteq \mathcal{H}_1$, $\mathcal{H}'_2 \subseteq \mathcal{H}_2$, whose sizes are at least $\frac{(c/5)^A}{330} n$, and every element of $\mathcal{H}'_1$ crosses all elements of $\mathcal{H}'_2$. Setting $\mathcal{S}_1 := \{s(h) : h \in \mathcal{H}'_1\}$, and $\mathcal{S}_2 := \{s(h) : h \in \mathcal{H}'_2\}$, the result follows. Case 3 can be treated similarly. $\square$

3 Proof of Theorem 4

The proof is based on a variant of Szemerédi's Regularity Lemma, which was discovered by Komlós (see [KS96]) and can be established by an elegant argument.

For any graph G and for any disjoint subsets $X, Y \subset V(G)$, let $E(X, Y) \subseteq E(G)$ denote the set of edges of G running between X and Y. Clearly, we have $|E(X, Y)| \le |X||Y|$. For any $\gamma, \delta > 0$, we call the pair (X, Y) (γ, δ)-*superregular* if for every $X' \subseteq X$ and $Y' \subseteq Y$ satisfying

$$|X'| \ge \gamma |X|, \quad |Y'| \ge \gamma |Y|,$$

we have

$$|E(X', Y')| \ge \delta |X||Y|.$$

Lemma 3. ([KS96]) *Let $\gamma > 0$ be a sufficiently small constant, and let $\delta > 0$.*

Then any graph with n vertices and at least δn^2 edges has a (γ, δ)-superregular pair (X, Y) with

$$|X| = |Y| \geq \delta^{1/\gamma^2} n.$$

First we establish

Theorem 8. *For every $\delta > 0$, there exists an integer $k = k(\delta) > 0$ with the following property. The intersection graph G of any system of n segments in the plane has k bipartite subgraphs, which altogether cover all but at most δn^2 edges of G.*

Proof. Set $G_0 := G$, and let β be a small positive constant to be specified later. Suppose that for some $i \geq 1$ we have already defined G_{i-1}. If G does not have a complete bipartite subgraph H_i, which contains at least βn^2 edges of G_{i-1}, then stop. Otherwise, pick such a subgraph H_i, and let G_i denote the graph obtained from G_{i-1} by the deletion of all edges belonging to H_i. Obviously, this procedure will terminate in

$$j \leq \frac{|E(G)|}{\beta n^2} \leq \frac{1}{2\beta}$$

steps, with a graph G_j.

We claim that G_j has fewer than δn^2 edges, provided that β is sufficiently small. Suppose that this is not true. Then, according to Lemma 3, G_j has a (γ, δ)-superregular pair (X, Y) with

$$|X| = |Y| \geq \delta^{1/\gamma^2} n,$$

where $\gamma := \frac{\delta^A}{330}$. Let $\mathcal{S}_X$ and $\mathcal{S}_Y$ denote the corresponding families of segments. In view of Theorem 7, there are two disjoint subsystems $\mathcal{T}_X \subseteq \mathcal{S}_X$ and $\mathcal{T}_Y \subseteq \mathcal{S}_Y$ with

$$|\mathcal{T}_X| = |\mathcal{T}_Y| \geq \frac{\delta^A}{330}|X| = \gamma|X| = \gamma|Y|$$

such that every segment in $\mathcal{T}_X$ crosses all elements of $\mathcal{T}_Y$. Let X' and Y' denote the subsets of X and Y, corresponding to $\mathcal{T}_X$ and $\mathcal{T}_Y$, resp. Then X' and Y' induce a complete bipartite subgraph in G. Furthermore, using the fact that (X, Y) is a (γ, δ)-superregular pair in G_j, we obtain that at least

$$\delta|X'||Y'| \geq \delta\gamma^2\delta^{2/\gamma^2} n^2$$

edges between X' and Y' belong to G_j. Therefore, if we choose β so small that this last quantity exceeds βn^2, then we could continue our procedure and define the next graph G_{j+1}. This contradiction completes the proof. □

Obviously, a similar result holds for $\overline{G}$, the *complement* of a segment intersection graph G.

Theorem 9. *For every $\delta > 0$, there exists an integer $k = k(\delta) > 0$ with the following property. The complement $\overline{G}$ of the intersection graph G of any system of n segments in in the plane has k bipartite subgraphs, which altogether cover all but at most δn^2 edges of $\overline{G}$.*

Now we are in a position to prove Theorem 4. Let G denote the intersection graph of $\mathcal{S}$. Let δ be a small positive constant which will be specified later. By Theorems 8 and 9, there is a family

$$\mathcal{F} = \{A_1, B_1, A_2, B_2, \ldots, A_{2k}, B_{2k}\}$$

of subsets of $V(G)$ such that

1. A_i and B_i are disjoint $(1 \le i \le 2k)$;
2. $A_i \times B_i$ is contained either in $E(G)$ or in $E(\overline{G})$ $(1 \le i \le 2k)$;
3. all but at most $2\delta n^2$ pairs $\{u, v\} \subset V(G)$ are covered by $\cup_{i=1}^{2k} A_i \times B_i$.

We say that two vertices of $V(G)$ are of the *same type*, if every member of $\mathcal{F}$ contains both or neither of them. The number of different types is at most 3^{2k}. A given type is *negligible*, if fewer than

$$s := \frac{\varepsilon n}{3^{2k}}$$

vertices have it. Letting V_0 denote the set of all vertices with negligible types, we have $|V_0| < \varepsilon n$.

Divide the elements of $V(G) - V_0$ into groups $V_1, V_2, \ldots, V_K$ of almost the same size: for every $1 \le i \le K$, let $s \le |V_i| \le 2s$. Clearly, we have

$$\frac{(1-\varepsilon)n}{2s} \le K \le \frac{n}{s}.$$

A pair (i, j), $1 \le i \ne j \le K$ is called *exceptional*, if $V_i \times V_j$ is not contained in $\cup_{i=1}^{2k} A_i \times B_i$. For every non-exceptional pair (i, j), V_i and V_j induce a complete bipartite subgraph either in G or in $\overline{G}$.

Let m denote the number of exceptional pairs. The total number of pairs $\{u, v\} \subset V(G)$ for which $u \in V_i$, $v \in V_j$ for some exceptional pair (i, j) is at least ms^2. On the other hand, by condition 3 above, this number cannot exceed $2\delta n^2$. Thus, we obtain that

$$\frac{m}{K^2} \le \frac{2\delta n^2}{s^2 K^2} \le \frac{2\delta n^2}{s^2} \frac{4s^2}{(1-\varepsilon)^2 n^2} = \frac{8\delta}{(1-\varepsilon)^2}.$$

This is smaller than ε, if δ is sufficiently small, so the partition of $\mathcal{S}$ corresponding to $V_0 \cup V_1 \cup \ldots \cup V_K$ meets all the requirements of Theorem 4.

4 Concluding Remarks

First we show how Theorems 5 and 6 follow from the previous results.

Proof of Theorem 5. Let G be a geometric graph with n vertices and at least cn^2 edges. The next result of Ajtai, Chvátal, Newborn, Szemerédi [ACNS82] and, independently, Leighton [L83] (see also [PA95], [PT97]) implies that there are at least $\frac{c}{64}e^2$ crossings pairs of edges.

Lemma 4. *Let G be a geometric graph with n vertices and $e > 4n$ edges, for some $c > 0$. Then G has at least $\frac{e^3}{64n^2}$ crossing pairs of edges.*

Thus, we can apply Theorem 2 to the system $\mathcal{S} = E(G)$. We obtain two subsets $E_1, E_2 \in E(G)$ such that every edge in E_1 crosses all edges in E_2, and $|E_1| = |E_2| \geq \frac{(c/32)^A}{336}cn^2 > (c/32)^{A+2}\binom{n}{2}$. □

Theorem 6 can be proved similarly. The only difference is that, instead of Theorem 2 and Lemma 4, we have to use Theorem 3 and

Lemma 5. ([P91]) *Let G be a geometric graph with n vertices and $e \geq 3n/2$ edges, for some $c > 0$. Then G has at least $\frac{4e^3}{27n^2}$ pairs of edges that do not cross and do not share an endpoint.*

The above theorems can also be established using Szemerédi's Regularity Lemma [S78]. However, then the dependence on c of the sizes of the homogeneous subsystems whose existence is guaranteed by our results gets much worse.

According to an old theorem of Kővári, Sós, and Turán [KST54], every graph with n vertices and at least cn^2 edges has a complete bipartite subgraph with $c' \log n$ vertices in its classes, where $c' > 0$ is a suitable constant depending on c. This immediately implies that Theorem 2 holds with the much weaker bound $c' \log n$ instead of $c'n$.

For some computational aspects of recognizing intersection graphs of segments, see [KM94].

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3–Dimensional Single Active Layer Routing

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Abstract. Suppose that the terminals to be interconnected are situated in a rectangular area of length n and width w and the routing should be realized in a box of size $w' \times n' \times h$ over this rectangle (single active layer routing) where $w' = cw$ and $n \leq n' \leq n + 1$. We prove that it is always possible with height $h = O(n)$ and in time $t = O(n)$ for a fixed w and both estimates are best possible (as far as the order of magnitude of n is concerned). The more theoretical case when the terminals are situated in two opposite parallel planes of the box (the 3–dimensional analogue of channel routing) is also studied.

1 Introduction

Traditionally, the detailed routing phase of the design of VLSI (Very Large Scale Integrated) circuits was considered as **a 2-dimensional problem,** gradually extended to $2, 3, \ldots$ layers. Even within this problem single row routing and channel routing (where the terminals to be interconnected are on one side, or on two opposite sides, respectively, of a rectangle) are the better understood subproblems, where the inputs are essentially one-dimensional (one or two lists of terminals, of length n, also called the *length* of the channel). Here the main aim is to realize the routing, and since its 'horizontal' size is given, its 'vertical' size, or *width*, w should be minimized. An important quantity is the *density* of the problem: the maximum number d so that there exists a vertical straight line cutting d nets into two.

In case of the 2-layer Manhattan model

- (1) single row routing is always possible in $O(n)$ time with $w = d$, see [12, 17],
- (2) channel routing is **NP**-hard [18, 26],
- (3) but this latter becomes always possible in $O(nd)$ time with $w = O(d)$ if we may extend the length of the channel by introducing additional columns [3, 6, 14]. In fact, the number of these additional columns may be as large as $O(\sqrt{n})$, see [3, pp. 212–213].

In spite of (2), there are plenty of practically effective algorithms available, which can solve 'difficult' problems (of length 150...200) with width around 20 [17, 10]. In case of switchbox routing (where the terminals are on all the four sides of a rectangle – a real 2-dimensional problem) instances with length 23 and width 15...16 are already 'difficult' problems on 2 layers [5, 8, 15, 22].

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If more than two layers are permitted, both channel and switchbox routing become easier, see for example [7] and [4, 25], respectively.

As technology permits more and more layers, a 'real' **3-dimensional** approach becomes reasonable. There are plenty of deep results in this area, see [1, 2, 9, 11, 13, 19, 20, 21, 23, 24], for example. Most of them embed certain 'universal-purpose' graphs (like n-permuters, n-rearrangeable permutation networks, shuffle-exchange graphs) into the 3-dimensional grid, ensuring that *pairs* of terminals can be connected, moreover, in some papers along *edge-disjoint* paths. Our result below is of much simpler structure but it allows *multiterminal nets* as well, and ensures *vertex disjoint* paths (or Steiner-trees) for the interconnections of the terminals within each net.

Throughout, except in the last section, we restrict ourselves to the *single active layer case* (all the terminals are on a single plane and the third dimension (above this plane, with *height* h) is for interconnections only). The terminals occupy certain gridpoints of an $n \times w$ rectangle. Henceforth we will use 'vertical direction' to refer to the direction of h (that is, to the direction perpendicular to this $n \times w$ rectangle) and not for the direction of w.

One can easily see even in small instances like 4×1 that a routing is usually impossible unless either the length n or the width w may be extended by introducing extra rows or columns between rows and columns of the original grid (compare with (3) above). If it is allowed to introduce both extra rows and extra columns then there is a trivial upper bound of $h = O(wn)$ for the height of the routing; see Lemma 1 below.

Our main result is that if w is fixed and n becomes large, a routing of height $O(n)$ can be attained not only in the aforementioned trivial way, but even if the new length n' satisfies $n \leq n' \leq n + 1$ and only w is extended to $w' = cw$, where c is a suitably chosen constant (we will show that $c \geq 8$ suffices). In view of Lemma 2 this linear bound is best possible. Moreover, our algorithm realizes this in $O(n)$ time, which is also essentially best possible.

Throughout this paper we are going to think of w as fixed and try to obtain bounds for the height as a function of n only.

2 Definitions and Main Results

The vertices of a given (planar) grid of size $n \times w$ are called *terminals.* A *net* N is a set of terminals. A *single active layer routing problem* is a set $\mathcal{N} = \{N_1, N_2, \ldots, N_t\}$ of pairwise disjoint nets. n and w are the *length* and the *width* of the routing problem, respectively.

By a *spacing of s_w in direction w* we are going to mean that we introduce $s_w - 1$ pieces of extra columns between every two consecutive columns (and also to the right hand side of the rightmost column) of the original grid. This way the width of the grid is extended to $w' = s_w \cdot w$. A *spacing of s_n in direction n* is defined analogously.

A *solution with a given spacing s_w and s_n* of a routing problem $\mathcal{N} = \{N_1, N_2, \ldots, N_t\}$ is a set $\mathcal{T} = \{T_1, T_2, \ldots, T_t\}$ of pairwise vertex–disjoint Steiner–

trees in the cubic grid of size $(w \cdot s_w) \times (n \cdot s_n) \times h$ (above the original planar grid containing the terminals) such that the terminal set of T_i is N_i for every $1 \le i \le t$. h is called the *height* of the routing.

Since w is fixed, one can also imagine the input as w *rows* of terminals (each of length n) or as a set of $\binom{w}{2}$ channel routing problems, each with length n and with a given density. Let D be the maximum of these densities. Clearly, $D \le n$.

Theorem 1. *If $s_w \ge 8$ then for any fixed value of w and for any n a single active layer routing problem can always be solved in time $t = O(n)$ and with height $h = O(n)$ such that the length n is preserved or increased by at most one. Both linear bounds are best possible.*

Our algorithm gives $t = O(w^3 n)$ and $h = O(wD)$.

3 Straightforward Bounds

Lemma 1. *If $s_w \ge 2$ and $s_n \ge 2$ then every routing problem can be solved with height $h \le \frac{w}{2}n$.*

Proof. We assign a separate layer to each net. For every terminal we introduce a vertical wire segment to connect the terminal with the layer of its net. The interconnection of the terminals of each net can now be performed trivially on its layer using the extra rows and columns guaranteed by the spacing in both directions.

Since 1–terminal nets can be disregarded, the number of nets is at most $\frac{1}{2}nw$ thus $h \le \frac{w}{2}n$ follows immediately.

Lemma 2. *For any given n there exists a routing problem that cannot be solved with height h smaller than $\frac{n}{2s_w}$.*

Proof. Let, for simplicity, the width and the length be even, let $w = 2a$ and $n = 2b$. Consider the following example (the idea is very similar to those in [4, 16]). Suppose that each net consists of two terminals in central-symmetric position as shown in Figure 1.

The number of nets is an. Since each net is cut into two by the central vertical line e, any routing with width $w' = s_w w$ and height h must satisfy $w'h \ge an$. Therefore $h \ge (w/2w')n$, hence $h \ge \frac{n}{2s_w}$.

Since in the above example $D = n$, this also proves the lower bound $h = \Omega(D)$.

The straightforward lower bound for the time is the length of the input, that is, $t = \Omega(wn)$.

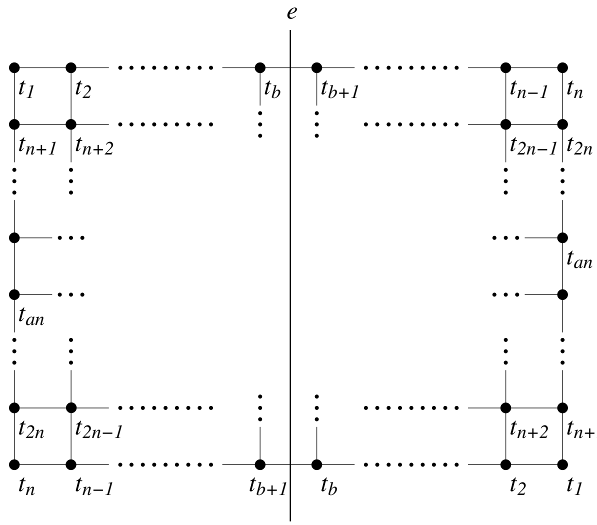

Fig. 1.

4 Two Simple Steps

1. Suppose at first that $w = 1$. Then what we have is essentially a single row routing problem with density d. Each net determines an interval of length at most n and these intervals can be packed in a vertex-disjoint way into d parallel lines, usually called *tracks*, using the 'left-edge algorithm' [12, 17]. Using the classical 2-layer Manhattan model, we can arrange the tracks in a horizontal plane, as shown in the top of Figure 2, thus realizing a routing with $w' = d$ and $h = 2$. However, alternatively these tracks can occupy either a vertical plane, leading to $w' = 2$ and $h = d$, or two vertical planes, leading to $w' = 3$ and $h = \lceil d/2 \rceil$, see the middle and the bottom drawing of Figure 2, respectively. (Theoretically one can pack the tracks to more vertical planes and thus ensure $h = \lceil 3d/(2w') \rceil$ for larger values of w' as well but it does not seem to be interesting.) Throughout in Figures 2, 3 and 6 continuous lines denote wires while dotted lines are for the indication of coplanarity only.

Similarly, if $w = 2$ then we have a channel routing problem with density d and using the same linear time algorithm we can always realize a routing with $w' = d + 1$ and $h = 3$ or with $w' = 3$ and $h = d + 1$, see Figure 3. This method is also well-known, it dates back at least to [7].

The right hand side of Figure 3 shows the essential idea of our algorithm: immediately at the level of the terminals (when leaving the 'single active layer') we increase the width from w to $w' = s_w w$ in order to make enough space 'between the rows of the terminals' for the vertical plane(-s) containing the interconnecting wires. The same process is illustrated for $w = 6$ in Figure 5 below.

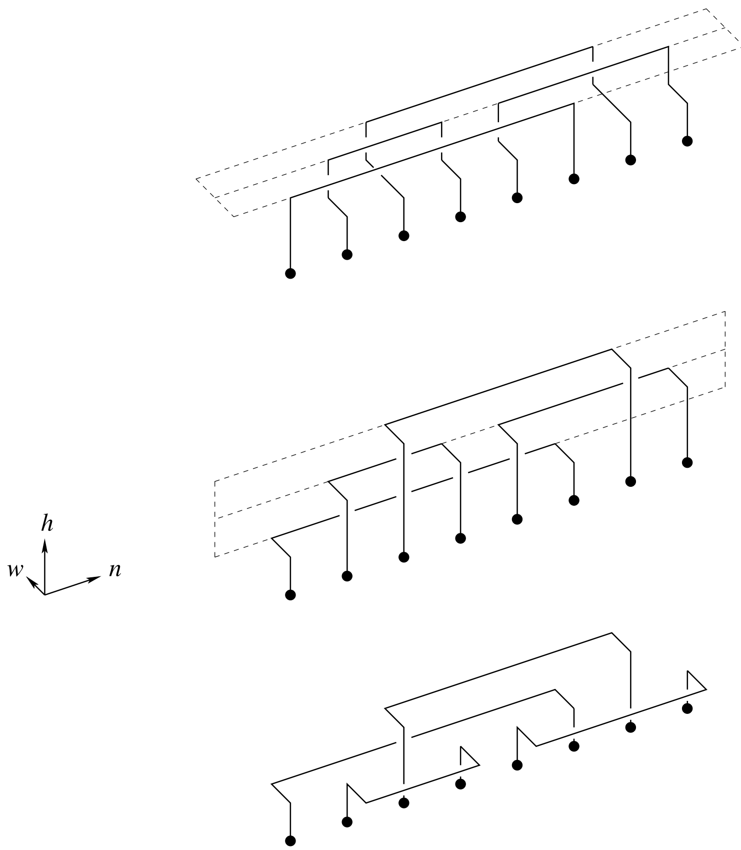

Fig. 2.

Later we shall need the following observation: if a larger horizontal area is available, we can decrease the height to $h = 2\lceil d/(w' - 2)\rceil + 2$. For this we can arrange the tracks in $\lceil d/(w' - 2)\rceil$ parallel horizontal planes – however, we need an 'empty' plane between two consecutive planes of tracks to ensure that the endpoints of the intervals can always reach the terminals, even if they are 'in the wrong side', as terminals t_2 and t_3 in Figure 3. Therefore the basic quantity hw' can be upper bounded by essentially $3d/2$ for $w = 1$ but only by $2d$ for $w = 2$.

2. Let us turn now to the general problem with width w. Since the terminals occupy certain gridpoints of an $n \times w$ rectangle, we consider them as a collection of w parallel *rows,* each of length n. We wish to solve $\binom{w}{2}$ channel routing problems one after the other. Figure 4 illustrates this for $w = 6$. At first (going from the bottom of the figure to the top) we solve those $w - 1$ channel routings where the rows are adjacent (first 'floor'), then those $w - 2$ ones where the distance of two rows is two (second 'floor') etc.

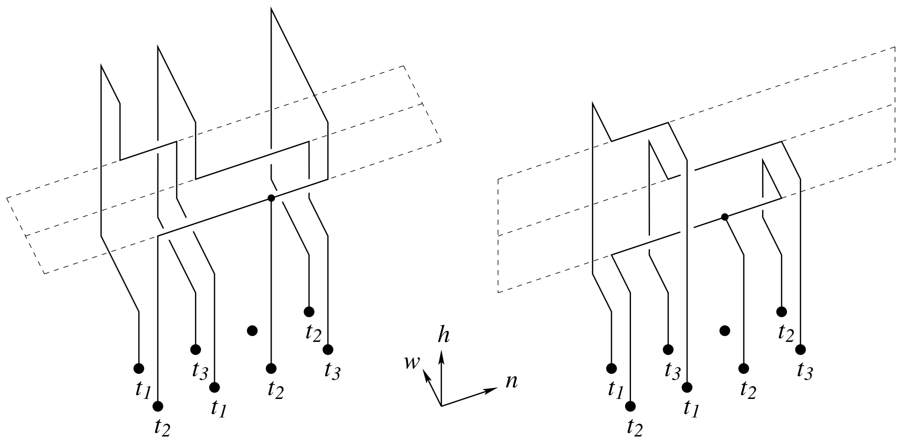

Fig. 3.

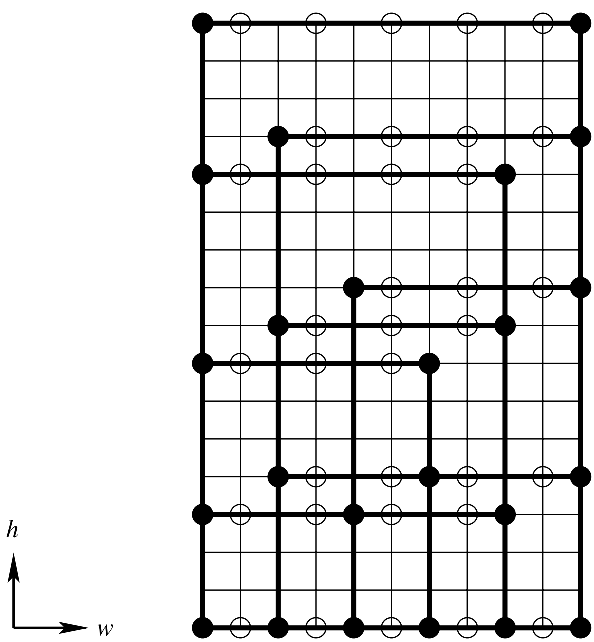

Fig. 4.

The $w-1$ channel routings at the first floor (actually the $w-1$ vertical planes containing the wire segments of these channel routings) do not interfere with one another (these are illustrated by the $w-1$ empty dots in the bottom horizontal line of the figure).

On the other hand, the $w-2$ channel routings at the second floor do not have this property hence this floor will have two levels, one for the row pairs 1 and 3, 3 and 5, etc and one for the row pairs 2 and 4, 4 and 6 etc. In general, floor f contains l_f levels where $l_f = f$ if $1 \leq f \leq \lfloor w/2 \rfloor$ and $l_f = w - f$ if $\lfloor w/2 \rfloor < f \leq w-1$. Solid dots illustrate the rows, the vertical lines in the figure show that the terminals within a row may appear at different floors. Empty dots indicate the areas where these two rows can be interconnected. Hence such an empty dot may indicate a contribution of at most D to the final height (compare with the right hand side of Figure 3).

Observe that there are two empty dots between two solid ones in the second floor, three empty dots between two solid ones in the third floor etc. Hence the total height requirement is *not* $D \times \sum_{f=1}^{w-1} l_f = O(w^2 D)$ but only $2D \times \sum_{f=1}^{w-1} \frac{1}{f} l_f = O(wD)$. The extra constant 2 is due to the necessary empty planes between the consecutive planes of tracks, as explained in the last remark in Step **1** above, concerning the empty plane between the consecutive planes of tracks.

In the introduction we mentioned that the width of the input must be extended to $w' = s_w w$. Figure 4 might give the wrong impression that $s_w = 2$ suffices. However, as we shall see in the next section, the realization of the 'crossings' in the figure requires much more space, leading to $s_w = 8$.

5 The Real Routing

For future reference we are going to introduce the following terminology. By a *w–plane* we are going to mean a plane that is perpendicular to the width of the routing, that is, to the 'vector' w of Figures 2, 3 and 6. Analogously, *h–planes* and *n–planes* are planes perpendicular to the height and the length of the routing (or to the vectors h and n), respectively. Similarly, by a *w–wire segment* we are going to mean a wire segment that is parallel to the width of the routing or to the vector w. *h–wire segments* and *n–wire segments* are defined in the same way. (Note that for example an h–plane is a horizontal plane, while an h–wire segment is a vertical wire segment.)

In the previous section Figure 4 illustrated the order how the $\binom{w}{2}$ channel routing problems are routed one above the other. (Of course it is possible that the terminals in row i and those in row k do not share any net and therefore a whole level within a floor is missing.)

However, Figure 4 may alternatively be considered as a 'cross-section' of the routing by an n–plane of size $w' \times h$ (and then there are n copies of these cross-sections, one behind the other). In this sense an empty dot in a particular level indicates a whole w–plane of length n and of height at most $2\lceil D/f \rceil$ containing several wires one above the other for that channel routing. Hence a horizontal line in Figure 4 between a solid and an empty dot indicates a wire segment going towards this plane – but it may go for a wire segment running in this plane or it may wish to avoid it and go for a wire segment in one of the other parallel planes (that is, towards one of the further empty dots).

It is very important to realize, therefore, that there are two types of 'crossings' which have to be avoided if we wish to realize the final 3-dimensional routing along the n–planes one after the other:

– Type 1 — A vertical line, connecting two solid dots, and a horizontal line, connecting two empty dots, may cross each other in Figure 4.
– Type 2 — A horizontal line which passes through an empty dot in Figure 4 may, in fact, not use that particular n–wire (which is perpendicular to the actual n–plane).

The basic point is that crossings of Type 2 can be avoided by a detour within the actual n–plane (increasing the height by one and the width by two) but crossings of Type 1 can be avoided outside the n–plane only. Therefore we must use the adjacent n–plane as well. But what happens if in this latter n–plane there is another vertical line interconnecting two solid dots (or in the real routing: a h–wire segment coming from a terminal) blocking the detour?

We avoid this problem in the following way: Since $w' > w$, we must, in any case, start the routing in each n–plane by 'expanding' the w terminals into larger distances. This increases the height by $w/2$ and can be performed like in either of the ways shown in Figure 5. Now, if we use the two kinds of expansions alternatively then the terminals within a single row will form a zigzag pattern and hence two h–wires that 'should' block each other in two consecutive n–planes, will actually be shifted by two units away.

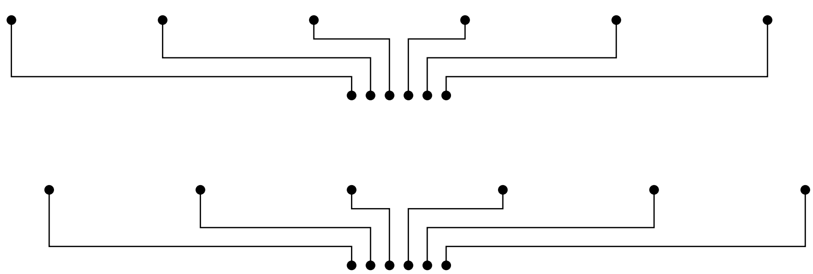

Fig. 5.

This way we have ensured that a Type 1 detour will not be blocked by an h–wire segment coming from a terminal. However, one difficulty remains to be solved: A Type 1 detour can still cross a w–wire segment that goes in the same h–plane one unit behind (connecting two w–planes within a level). In order to handle this, let us number the n–planes from 'front' to 'back' with the numbers $1, 2, \ldots, n$ and the h–planes form 'bottom' to 'top' with the numbers $1, 2, \ldots, h$. Now let us declare the following rule: If a w–wire segment goes in an n–plane numbered with an even number, then it must go in an h–plane also numbered with an even number; similarly, if a w–wire segment goes in an n–plane numbered with an odd number, then it must go in an h–plane also numbered with an odd number. Since the height of a level is always the double of the number of tracks in it, the above rule can obviously be fulfilled. (This way some of the Type 2 detours will become unnecessary: If a w–wire segment reaches a w–plane (containing some of the tracks within a level) between two consecutive tracks, then the w–wire segment can cross the w–plane without meeting the tracks, there is no need for a Type 2 detour.)

Figure 6 illustrates most of these situations in a single drawing. Recall that continuous lines are wires, dotted lines are for the indication of coplanarity only. Observe that $t_1, t_2, \ldots$ are terminals within a single row (illustrating the aforementioned zigzag pattern), while the terminals $t'_1, t'_2, \ldots$ form the next row. It might be instructive to recall that the detour between A and B is of Type 1 while that between C and D is of Type 2.

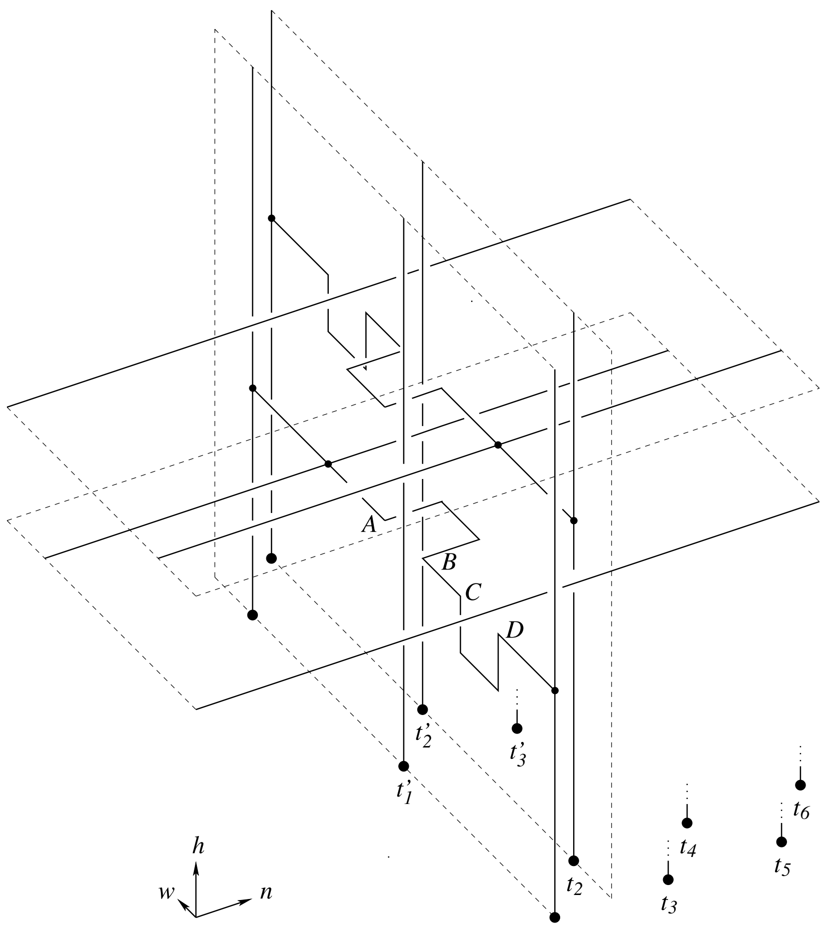

Fig. 6.

Figure 7 shows a part of Figure 6 again in order to explain why s_w needs to be as large as 8.

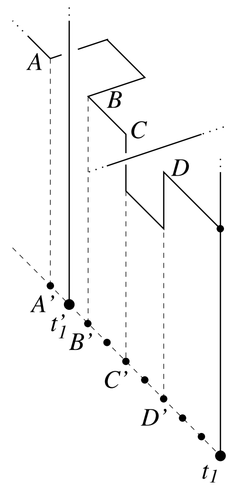

Fig. 7.

6 3–Dimensional Channel Routing

Throughout this paper we have dealt with the single active layer routing problem, which is, in a sense, the 3–dimensional analogue of single row routing. However, the 3–dimensional analogue of channel routing may also be of interest not only from a technical point of view (see [11] for example), but also in a theoretical sense: in contrast to the essential difference in complexity between single row routing and channel routing in the 2–layer Manhattan model (see (1) and (2) in the Introduction), there does not seem to be such a difference between their 3–dimensional analogues. The *3–dimensional channel routing problem* is defined as two parallel rectangular planes of size $n \times w$ containing all the terminals to be interconnected (by vertex disjoint Steiner trees) in a box of size $n' \times w' \times h$ between the two parallel planes. As before, we suppose that w is fixed and n can be very large hence we allow $w' = s_w w$ but $n \leq n' \leq n + 1$ only.

Theorem 2. *For any fixed value of w and for any n such a 3-dimensional channel can always be routed in $t = O(n)$ time such that the length n is preserved or increased by at most one, the width is extended to $w' = s_w w$ and the required height is $h = O(n)$. Both linear bounds are best possible.*

The lower bound is the same as in Lemma 2. The routing can be performed basically along the same line as explained in Sections 4 and 5 for the single active layer case. Just like the two 'expansions' of Figure 6 are alternating along the direction n, the expansions opposite to each other on the two parallel planes should also be shifted by two units away.

Acknowledgments

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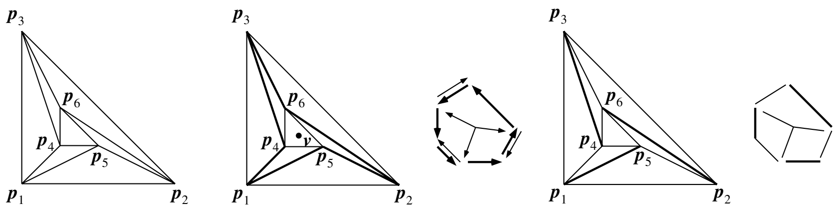

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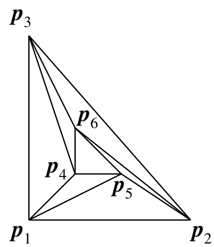

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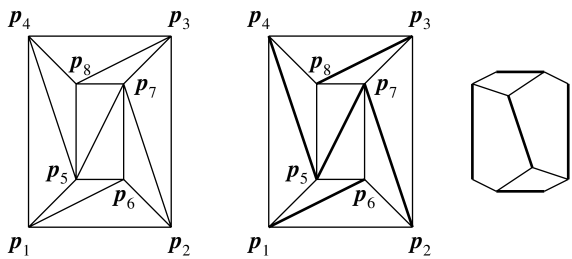

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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 339-350, 2001.

rese r ers en n r en r n e r e i n e p g n n r is gi en i in r es e *r s r h pr l* e e in r er s e ei er ep n e gi en r r e i e r g i ee *l* g e n i e n sp e s i n e r se r pr e n s e i n p g n r se r e se r er is se r e se r er e er eir r eri i ns res e p i e n n nifie i e e r eri i n r se r ers iffers r r se r ers

n is p per e gi e n nifie n e ien s i n e r se r pr e r se r er e presen n p i g i e n sp e g ri r gener ing se r s e e i i e is s ere is e n er er i es n $\leq$ is e ini n er se r in s r i ns re ire e r e p i i r g ri is ine i en i ing ri i isi i i e en s rre in n e p sing e se r s e e se n e r er re r e n e si e e en e s e e pr e se r ing r se r er e e ensi n is se n gener i i n e n i n isi i i in isi i i

y

e i firs gi e si efini i ns r e r se r pr e n en re ie e e n n g r pr e

ic d finiti n

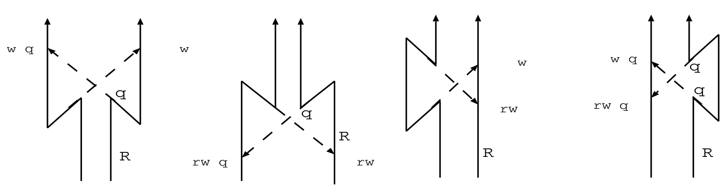
A r is si p e p g n i e ing n se fin erse i ns n r es i r n i s n r r n enien e e ss e e r is in gener p si i n in e p ne is n ree er i es re ine r n n ree e ge e ensi ns e n p in p in s x re s i e *s l* i e ine seg en $\overline{x}$ nne ing e is en ire n ine in r regi ns R e s R is *w ly s l* r i e er p in in R is isi e r s e p in in

e *s* en e e p si i n se r er n e p si i ns en p in s is s ig s n e n r i e respe i e A p in x is s i e *d* *d* r *llu* *d* i e i x is n ine in ne e ine seg en s *s* *s* An regi n ig n in e in r er i e se p si i n is n n n e se r er s e is p e ing r i r ri s is s i e *d*; er ise i is s i e *l r* A *s r h s h dul* e se r er r is p e *S* *s* n in s n i ns *s* : s e in r er is e e s ne *s* *s* i ing i e i r g is s i e se r e i ere e is s se r s e e e se r er r

rd i ity c rrid r

A rri r is si p e p g n i n en r n e n n e i *v* n i s n r e g r pr e s s i ere is in s per i e g r s n e ng p g n ins n R rien e r v ne ise n e er n er ise in s n re s isi e r p in s e s *pr d s* n *su ds* i e en n er e re en r ersing r *s* e ri e e efini i n r R is s e ri e n en e e ing n i ns g r s n n n R respe i e A w l in p g n n en e r e s p ir n in s n i ns : n : R ere v n x n x re isi e r x An ine seg en $\overline{x\ x}$ is e $w\ l\ s$ A s $w\ l$ r $w\ l\ s$ $\overline{q}$ h $h\ r\ \overline{q}$ ere n q q s fi e n i i ns q n q e is $s\ r$ h i n re n ne n i ns r er e x p g n in e $S\ cc\ x$ en e e ere e in i e i e s ee ing x n e x e ere i e i e pre e ing x A er e is r i i s in eri r ng e is gre er n ; er ise i is An i p r n efini i n r re e er i es is *r y sh s*: e r r s r re e ere in r R en e B *w r)* is e firs p in i e s in e ire i n r $S\ cc$ n e r r r s *rw r)* is e firs p in i e e s in e ire i n r *e* ee ig e efine e rien i n e seg en c r s r c r

A p ir re e er i es q R is s i r *d dl* r n e r s r i q c R n c q ig r *d dl* v r v e r s r i q R n q ig A p ir re e er i es resp q q R is s i r *w d* i n c in R resp q q n q c q in ee igs

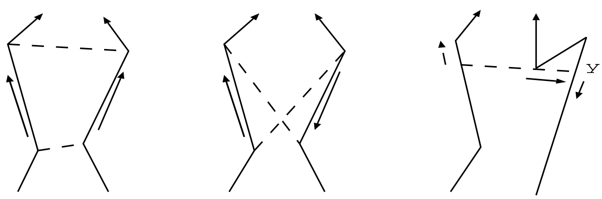


g 1 a l cks a w s

A rr d r s w l l d ly h h s d R r u u lly w ly s l d d dl s ur ur h r r h d h h r r w d s r R s dd d rr d r s s r h w l l I s s h w w l l y rr d r u pu s r h w l d g u pu w l wh r $\leq$ *s h l u r s r h s ru s*

o y o oo y -

A se r ins r i n se r er n e es ri e s ne e ing e e en r i ns : e se r er s n e en p in is s ig e ng seg en s sing e e ges s i n in erse i ns r ng ine seg en s $\overline{s}$ ring e e en ii ine seg en s in erse e er n iii ps r p in x i s e re e er e n e n r e er p in s e r e een s n is e en e ee ig

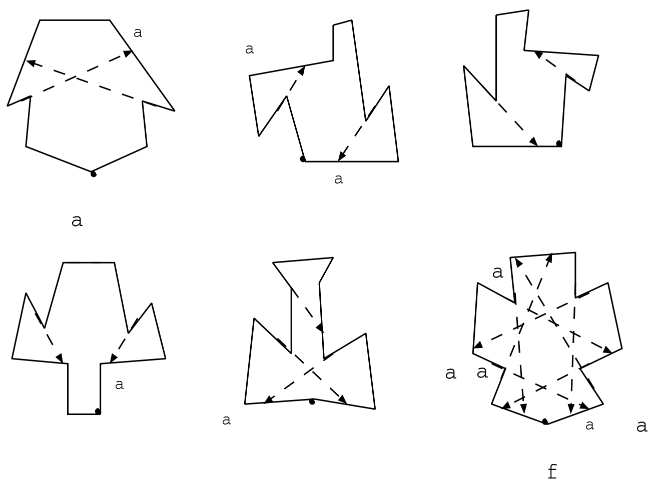


g S arc s r c s a 1 s arc r

s n e n in s n e n r p g n r se r ins r i n i r ii i e es is n in s n e n r r se r ins r i n iii e firs ins r i ns se r er re e r g r s i s n re nsi ere s g r s n e s ne is n e r e ins r i ns iii is e r e s ig e r s ping er en i e e rner e $\overline{x}$ in ig is e re n r d e re n er e ins r i n iii is e respe i e regi n e es n in e r e se n r re i e i is re erre s r d n ins r i n iii s e e n ins r i n ii s s n e rr s in ig r si p i i e re er *shl h* r s n ins r i n ii r iii r se n in s ins r i ns ii n r iii

n e rigin p g n se r pr e i is s efine i e se r er s ps r p in x e er n e n r As e i see in r r eri i n ins r i ns i n e p n r se r er e r p g n r is e i i n ins r i ns i es n ffe e rre ness e ne essi e re n e s ien e re is pr e i ins r i ns i n eres ing e en ins r i ns iii nee n' e nsi ere r e rri r se r pr e

e gi e n ern e r eri i n e se r e r s r r eri i n is n n ns r i e i e s n p i se r s e e s nifie i n e si e e en e r e se r e r s e pr p i i e gener e s e e n erning e ges n s r ig s is i e r pr is si p er n se gi en in e s r er e p in s n e p g n n r ise s n e n r s r ing e r r p e e r ering e nsi er s p in s $_l$ n s $_l$ ≤ ≤ r p in s in e n r i i r efini i ns n en e gi en s se in e i n r er e x S cc x en es e er e s ee ing x n e x e er e i e i e pre e ing x r re e er e e r n r r r s s c n re e firs p in s i e e s s in e ire i ns r S cc n r e respe i e An g s e ge r e e g ig n e efine in e e p in s n e n r re n r ere r $_l$ e ine i ies r e ges n e s s e r ing nge r i r e re er n e r n p in n e n r s e e i rs en is nsi ere s e r



g c s 1 a

r *A p ly l r s 1 s r h l h ll w d s s ru*

) h r r w r r s v d v su h h v c v v v) v c v v c v) v v v v) r v v v v d)

) h r r hr r r s v v d v su h h v v rs s w h v c v d v v c v v r v v v v)

) h r r w r pl s d s d ≤ ≤ y d l) su h h) d r S cc d dl d c) d r e d dl d d **c***) ll r s v w d h v d dl)*

r r e firs ern i e e n i i n ig e ere e es n in e en e ere $S\ cc\ v$ r $e\ v$ is e re r e i e pr see in e e ere $S\ cc\ v$ r $e\ v$ s e e re n e n s ess se r s e es s r ing r e si i ns s n in ig n e e i n g s

r e n i i n e n e si fin ree er i es e g $e\ v$ $S\ cc\ v$ n $S\ cc\ v$ in ig e s n p in e s r es p e een n er i es is isi e e ir en s r e re e r is n se r e

r e n i i n $S\ cc$ s e e re s ng e $S\ cc$ n $S\ cc$ n e s e e re s ng e e n $S\ cc$; er ise e ere ere n in e r er re sin e er i es v e een n e v e n s ess se r s e es e is ig

in n e e e re is r e e en ins r i ns i re en in nsi er i n

r e n i i ns n re essen i e s e s e n i i ns n gi en in e re e e n i i ns gi en in re s e re n n

e s n efine ri i isi i i e en s rre in e r sing ese e en s e n e e p n p i se r s e e r i i e is s e en e re e er e g n is i i e in pie es e en s ei er e ge in i en n i i i s e p g n n r A C is $s\ l\ y$ u i i pr es n e ng e in e pie e n ining i is en e C n is se ei er c r s e nn si ne s re er C is ss l i C is n n ine in n er C ere C is e er isi i i e e re e er i es ere essen i s re efine e r l r s e n ep essen i s is i e se in e e n n n r e pr e

r $A\ p\ ly$ $l\ r$ $s\ 1\ s$ $r\ h$ l h d s h r 1 ppl s

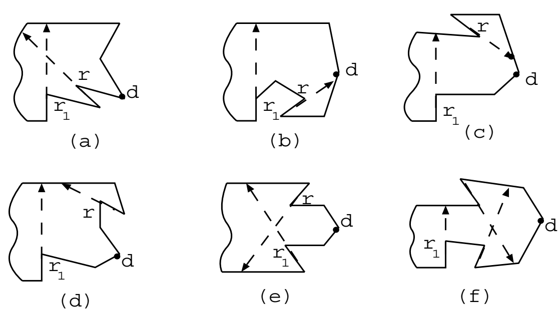
r Ass e e e r is in isi e r ; er ise n si p e e re e l e e se en e ri i er i es in e e in ise s n e n r s r ing e R en e e er en p in e essen i efine i ss gener i ss e is e rig e R se rien i n is efine s r R er ise is e e lR l n e se r s e e es ri e e r s n er ise

n e ing e en e x r $R\ x$ e in r x i x s x n r x re e er e in in s ne i s en er i es r eing isi e n p in in e pp si e in e i l er e

C s 1 h d s ru

C s 1 1 h d r s h r h ll ss l u s R ≤ ≤
ere e is s er e ere isi i i n essen i is s efine
is e e i s isi i i e en e e firs s er e see s
ig n ig 6 C e r e s ri i er e $_l$ pre e es i e $_l$
r se r s e e en s r A s e ss e e p si i n e
se r er s gi es e e en p in e r e n ing r e s ig
s ie e r

C s 1 1 1 irs ins $_l$ n R R re
e isi e er ise ere ere er ri i er i es e re ig
r e n i i n r ere r e ig e s r e een
$_l$ n R R e se e e is en e ; er ise e n i
i n r ere r e ig e en e e regi n R is e
g r s r se r e r er e R

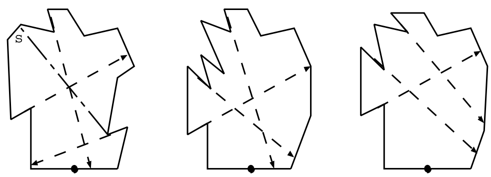


g as 1 1 1

C s 1 1 ≤ s r e n i i n e R
in erse s i R C nsi er firs e si i n ere e in
is e isi e R R R p in s in re isi e
e in erse i n p in R n R n ere is n re e
er e in R R R s c R
s e si p r e e s ig r R R ere
re s e er i es in R R R s c
R s e r i n is s ppe en e firs s er e is en
n ere ig e s en e e s pping p si i n e se r er e
i ere is s r ig r $\bar{s}$ in e is e firs er e s
is ing c R ere re n e ges in e in
R A s ere re n e ges in s ; er ise ere n ri i
r ere ere er ri i er i es e een n ere re n e
s e een ins s n R ; er ise e n i i n r
ere r e e in s is e isi e R ; er ise
ere n ri i ere ere er ri i er i es e een n r e
ing er e in s n s is e n i i n A s e
in R is e isi e s ; er ise ere ere e essen i
s s is e e e r ere n e firs er e s is ing

c R en e r i is pr e n e e r is e re sing s r ig r $\overline{s}$

n e si i n ere p in s in re n isi e e in er se i n p in R n R e e e firs re e er e in s e is in isi e in erse i n is i p ssi e r S cc e in isi e e in erse i n p in ; er ise e er e in s e ere ri i As es ri e e e s ig is en r e r R n s n ig r e e r is e re en s e ere s is ing c R is en n ere

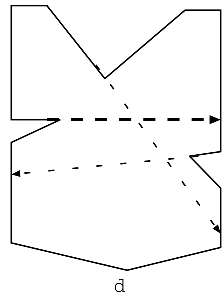


a

g as 1 1

C nsi er e si i n ere is n e isi e R R R e e e ing er e in se r r r s is e i ng se r e ing er i es ig in e R e seg en in erse s i R es n in erse i R e se e i e in is e isi e R R As ne e e s ig n e r e r R e si i n s n in ig nn r; er ise e n i i n ere r e e in is n e isi e R R e n erse is s r e; er ise e n i i n r ere r e e se e er i es n n e s r e een ins; er ise e n i i n r ere r e e s ig is en e r R sing

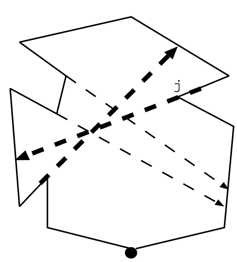
C s 1 1 h r $_l\overline{R}$ $_l$ s l r d e e regi n $_lR$ $_l$ is isi e e en p in $_l$ r R $_l$ i n s e r n e e re ing ne en p in in e er ng e n r $_lR$ $_l$ er ise e e e re e ere in e in r $_l$ R $_l$ s i s e r eing isi e R $_l$ n i s r r r s is e i ng se r e ing er i es ig 6 i s R $_l$ in e R R $_l$ is e isi e $_l$ e n r e e s ig r $_lR$ $_l$ e er e is n ine in e in r e i ere is s r ig r e se e i n e er e e in is e isi e R A s e



g as 1 1

in R is e isi e ; er ise e ing er e
in R ere ei er ri i r ge er i n s is
ere re n e s e een ins; er ise e n i i n
r ere r e ere re n e ges in e in r R ;
er ise ere ere er ri i er i es er $_l$ s e i is pr e
n e e r is e re is n n ine in e in r
e e r ing pr e re is repe e per r e n is
se r s e e r e ring r n e en e p

C s 1 h d r s h r h s $\overline{R}$ *d h l s* $\overline{R}$ *)* Ass e is e firs ri i er e s
is e e $\overline{R}$ e i ere is r ere
re n e s n r e s; er ise e n i i n r ere
s isfie e in R is e isi e $_l$; er ise n e
ing er e in R $_l$ s is e n i i n e in $_l$
is e isi e R ; er ise ere n ri i r e n i i n
ere r e en e e i is pr e n e e r is en e re
sing r

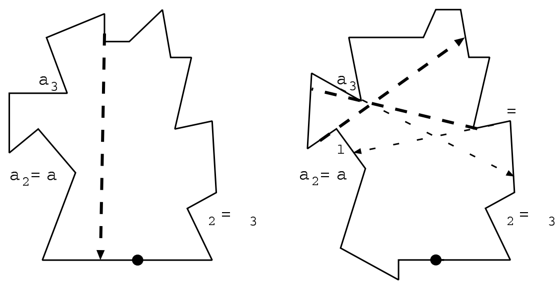


g as 1

e e s e r is p i e en e e s
ri i er e s is e rig si e $\overline{\quad R \quad}$ C e r se r s
en p in e een n en n s ess se r s e e r
er e v e een n is p i is is e se n e ge in e
in r $_l$ r r rs n r v; er ise
e n i i n ere r e ig n n e ge nn r in e in
r ; er ise resp ere n e s resp firs ri i er e
s is e rig resp e $\overline{\quad R \quad}$ resp $\overline{\quad R \quad}$

C s h d s ru is s e ri C se

C s B h d r ur in e **c** is n r e in is se
ere is e s er e s n rip e s is ing
pre e es n rip e s is ing s ee s n n
e s r ig



g as

Ass e firs ins $_l$ n R re e
isi e in e nei er e s n r e s r ere is r
e e s e r is p i is e rig
s e s n e e e er s en s in C se e n s
s e r is p i is e rig essen i
s en R r $\leq$ $\leq$; er ise $_l$ n R
re n $\overline{\qquad\qquad}$ e isi e As in C se n C se e e r e
regi n R in e r er e si i n s n in ig is
e er en n ere en ere is s r ig r $\overline{s}$ e e er
R n nei er e s n r rip es s is ing
e re r e si $\overline{\text{ i n s }}$ n in ig is ne er en n ere
en ere is s r ig r $_l R$ $_l$

e in $_l$ is n e isi e r R en R
is e isi e r $_l$; er ise e n i i n ere r e e
si i n ere R is n e isi e $_l$ n e e i
n g s i ss gener i ss e is e firs ri i er
e in $_l$ s e is in isi e n p in in R ig
en $_l$ e n R e re e isi e
e e s r; er ise s e rip es s is ing e

n i i n e re r en e ere is r *e*
in e *e* s e s e pr per s s s n e e r
e is p i

C nsi er e si i n ere ere is er e in *l* s S *cc*
is in isi e r n p in in R n is se e n ss e is
e rig essen i s Ass e s ere re S *cc* *l* e s;
er ise n si p e e e re n p i r S *cc* *l* As in
C ses n e e r e regi n *l*R *l* e si i n s n
in ig is e er en n ere in e ring *l*R *l* en R
er ise e n i i n ere r e n ere is s r ig r $\overline{s}$
pp se n *l*R *l* is e re is e e *l*R *l* en
ere is s r ig r *l*R *l* is e rig *l*R *l* s
in C se r e e s ig ise e r i n is s ppe en ei er
e e r is e re r re e er e ere n *c*
is en n ere n e er se e *s* en e e s pping p si i n e
se r er ig en ere is s r ig r $\overline{s}$ in e *l*R *l*
s e e re n e e se r s e e presen e ere is p i
n ses e e r is e re e s ins r i ns i re
ne er se

r *r p ly l r s p ss l r p l s r h s h dul r 1 s r h r g d sp s s wh r s h u r r s d* $\leq$ *s h l u r s r h s ru s r u r d l r*

r As s n in i es g i e es e se r i i p g n r A er i is erifie n ne n is r e e r n e ns r i e g ri presen e in e pr e re p se r s e e e e s e gener e s e e is p i is e s see e regi n R s e e re n e n e se r s e e gi en in C se e re is p i e si i n s n in ig is ne er en n ere in C se e se r s e e r e ring R is p i is is e se e se r g ri gi en ere is gree n e e si i n s n in ig is en n ere in C se s r ig pp ies i gi es p e e se r s e e i i r e se r g ri gi en in C se is p i e p i i e se r s e e in C se s e s in C se is re is sse in e pr e re p e es e pr

4 x o

r e r se r ing p g n r se r er n e e en e s e e r se r pr e r se r ers As s n in e e ensi n is se n gener i i n e n i n isi i i in isi i i e s i p r n is gener i e r s s in r s s in e e re gi en n ne r e p i n in r s s es i e n i ing e ins r i ns i nee n' e nsi ere r e r se r pr e e e e ing res

r *r p ly l r s p ss l r s r h s h dul r s r h r d sp s s*

r i e in is e en e s r

1 rass S z k a a as a S arc r a l r r a c rr r *l* 199) 97 41
as a S a all a wa a r c a l ar a w rks ac w s acl *EEE R b* 1997
as a S a all a wa s l as rs as a l al r *l* 1999) 471 49
4 r a al al r r w ar r l *l* 199) 1 44
ck a l w ar s r l *l* 199) 7 8
S a all S a Sl zk al r r s arc a l al r w a as l *16*) 9
7 S S a wa s l as rs as a l al r w a r *1* 1999) 81 9
8 S ark a wa S arc a l al r w r a 1 s arc r *l* **10**) 1
9 S z k a a as a S arc r l r rs a l al r **1** 199) 8 888
1 a c s l c rr r s arc r l c s Sc **1**)) 17
11 a S arc a s l l a s arc r r r s als a a s rac **1**) 14)
1 a ra a a a ak cr al al r r c s r c s r s wa c a r s *l* 199) 1
1 a ra a a a ak rr cr al al r r c s r c s r s wa c a r s *l* 1999) 19
14 a as a S z k a a a S arc r l r rs a l al r a r l s arc rs *1* 1997) 448 4

A New Structure of Cylinder Packing

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Abstract. We report a new periodic structure of the cylinder packing. All the cylinders are congruent and the length of the cylinders are infinite and their directions are restricted to only six directions of $\langle 110 \rangle$. Each cylinder is fixed by cylinders of other directions, so that the whole structure sustains mechanical stability. The packing density equals to $\frac{(351\sqrt{2}-108\sqrt{6})\pi}{1936}$ ($\simeq 0.376219$), which lies between two values ever known: 0.494 or 0.247. The arrangement of parallel cylinders forms a certain 2D rhombic lattice common to all of six $\langle 110 \rangle$ directions. Nevertheless the way of fixing cylinders is different in all of six directions: the cylinders of two directions are supported with the rhombus-type, and the cylinders of other four directions are supported with the equilateral-triangle-type. The structure containing the equilateral-triangle-type has never been known.

1 Introduction

The history of the research on the cylinder packing is much shorter than that on the sphere packing. Few mathematicians have treated the problem[1]. Some simple structures of the cylinder packing appear in the books about solid puzzles(e.g. Holden[2], Coffin[3]). O'Keeffe and Andersson applied the cylinder packing to the science[4][5][6][7]. They are crystal chemists and explained the garnet structure famous for its complexity by using a periodic cylinder packing restricted to only four directions of $\langle 111 \rangle$.

In engineering, some structures of the cylinder packing are used for the composite materials. Such structures are light and tough against the stress from various directions. Some periodic structures were designed (Hatta[8], Hijikata

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and Fukuta[9]). Stimulated by their researches, the authors investigated other periodic structures with six directions [10][11][12][13][14], and paid much effort exhaustively to find all the possible structures with high symmetry and density. More precisely, all the cylinders are supported by contacts of the cylinders in more than four directions among other five. Watanabe found the remarkable fact that each structure could continuously modify itself without changing its directions of cylinders, its packing density and its stability at the same time when the cylinders are kept tangent one another. An animation can be demonstrated on his web-site[14].

Quasiperiodic packings of cylinders were also reported. An architect Hizume extended Coffin's structure and Ogawa cooperated the research[15][16][13]. The structures have the icosahedral symmetry, and are composed of six directions indicated by $\langle 1\tau 0\rangle$ ($\tau = (1+\sqrt{5})/2$). Other quasiperiodic packings were discussed in recent papers[17][18][19].

The structures of the cylinder packing have to be researched more systematically. The present paper suggests further possibility of $\langle 110\rangle$ six-axes structures. The theory of $\langle 110\rangle$ structure will change to general one.

2 The New $\langle 110\rangle$-Structure

The structure is a six-axes periodic structure made of cylinders with one diameter d. Some similar structures whose packing density are 0.494 or 0.247 have been known and are listed on Table 1 in Appendix. For the sake of convenience, we call the structures respectively Type-I(density 0.494), Type-II(density 0.247), and Type-III(the present structure).

The six directions are $A(1,1,0)$, $B(1,-1,0)$, $C(1,0,1)$, $D(-1,0,1)$, $E(0,1,1)$, $F(0,1,-1)$. If we classify these directions according to perpendicularity, they separate into three groups: A–B, C–D, and E–F. Another classification is made by the relations of sixty degrees:

- B, D and F are sixty degrees each other. A common normal vector is $(1,1,1)$
- A, C and F are sixty degrees each other. Normal vector $(-1,1,1)$
- A, D and E are sixty degrees each other. Normal vector $(1,-1,1)$
- B, C and E are sixty degrees each other. Normal vector $(1,1,-1)$

When we care about cylinders parallel to A-direction in the structure of Type-III, they form a rhombic lattice on a plane perpendicular to A-direction. Each of other five directions also forms the rhombic lattice on a plane perpendicular to the direction. Moreover, the six rhombic lattices are congruent. A rhombus of the rhombic lattice has two diagonals, and we express that the length of the longer diagonal is $2a$ and that the length of the shorter diagonal is $\sqrt{2}a$. The relation between a and d is

$$a = (2/3)(1+2\sqrt{3})d. \quad (1)$$

The following expression is written with a and d to make formulae simple.

2.1 Equations describing the structure

For the description of the structure of cylinders, we use the line equation as the center line of the cylinder. Whether two cylinders contact each other or not is confirmed by comparing their distance with the diameter d. For Type-III, the equations of one cylinder about each direction are shown:

$$A_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ -\frac{a}{4} \end{pmatrix}, \tag{2}$$

$$B_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ \frac{a}{4} \end{pmatrix}, \tag{3}$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{2a-d}{2} \\ 0 \end{pmatrix}, \tag{4}$$

$$D_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{2a-d}{2} \\ 0 \end{pmatrix}, \tag{5}$$

$$E_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} \frac{a-d}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

$$F_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -\frac{a-d}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

where, t is a parameter and $-\infty < t < \infty$ for cylinders with infinite length.

All other cylinders A_{mn} of A-direction are obtained by parallel movements of A_1:

$$A_{mn} = A_1 + ma_1 + na_2 \tag{8}$$

where, a_1 and a_2 are vectors causing the parallel movements, and m and n are integers. In the same way, all the cylinders can be generated by parallel movements of one cylinder in other directions. All the vectors of the parallel movements are shown as follows:

A: $a_1 = (\frac{a}{2}, -\frac{a}{2}, a)$, $a_2 = (-\frac{a}{2}, \frac{a}{2}, a)$,
B: $b_1 = (-\frac{a}{2}, -\frac{a}{2}, a)$, $b_2 = (\frac{a}{2}, \frac{a}{2}, a)$,
C: $c_1 = (-\frac{a}{2}, a, \frac{a}{2})$, $c_2 = (\frac{a}{2}, a, -\frac{a}{2})$,
D: $d_1 = (-\frac{a}{2}, a, -\frac{a}{2})$, $d_2 = (\frac{a}{2}, a, \frac{a}{2})$,
E: $e_1 = (a, \frac{a}{2}, -\frac{a}{2})$, $e_2 = (a, -\frac{a}{2}, \frac{a}{2})$,
F: $f_1 = (a, -\frac{a}{2}, -\frac{a}{2})$, $f_2 = (a, \frac{a}{2}, \frac{a}{2})$.

An overview of this structure is shown in Fig. 1 and the projections to the planes perpendicular to $\langle 110 \rangle$-directions are in Fig. 2.

2.2 Packing Density

It is easy to calculate the packing density for the present periodic cylinder system. Because the cylinders don't overlap, whole packing density is six times as high as the density of one direction. The density of one direction is determined by the ratio of the area of a circle and of a rhombus. Hence, the whole packing density D is

$$D = \frac{\pi(d/2)^2}{(\sqrt{2}a)a} \cdot 6 = \frac{(351\sqrt{2} - 108\sqrt{6})\pi}{1936} \simeq 0.376219 \cdots . \tag{9}$$

The density lies between two values ever known: $\sqrt{2}\pi/9 \simeq 0.494$(Type-I) and $\sqrt{2}\pi/18 \simeq 0.247$(Type-II).

2.3 The Supported States

If we restrict the directions of cylinders to $\langle 110 \rangle$, there can be two typesof support, that is, the rhombic-type and the equilateral-triangle-type. In the rhombic-type, a cylinder is fixed by four directions which are sixty degrees against the cylinder. In the equilateral-triangle-type, a cylinder is fixed by two directions of sixty degrees and one perpendicular direction against the direction. Concerning the present structure, the cylinders of A- and B-directions are fixed with the rhombus-type and other four directions are fixed with the equilateral-triangle-type. The more detail is shown as follows:

- A-direction is fixed by C-, D-, E-, and F-directions
- B-direction is fixed by C-, D-, E-, and F-directions
- C-direction is fixed by A-, B-, and D-directions
- D-direction is fixed by A-, B-, and C-directions
- E-direction is fixed by A-, B-, and F-directions
- F-direction is fixed by A-, B-, and E-directions.

These are confirmed on Fig. 2. On the other hand, the supported state for Type-I and Type-II is shown in Fig. 3 and Fig. 4. Only the rhombus-type supporting is contained in Type-I and Type-II.

It is effective to investigate the structure about its arrangement along $\langle 111 \rangle$-directions. B-, D-, F-directions are perpendicular to $(1,1,1)$-direction, the fact mentioned at the begining of Sec. 2. Therefore the cylinders of the B-, D-, F-directions are stacked up along $(1,1,1)$-direction. In Type-III, the order of appearance along $(1,1,1)$-direction is $\cdots FBDFBDFBD \cdots$ and there is a gap $(= 0.43647d)$ between D and F. A list about all of four $\langle 111 \rangle$-directions is shown below, where the mark $\sqcup$ means the gap whose size is common to four directions.

- Along $(1,1,1)$: $\cdots FBD \sqcup FBD \sqcup FBD \sqcup FBD \cdots$
- Along $(-1,1,1)$: $\cdots CAF \sqcup CAF \sqcup CAF \sqcup CAF \cdots$
- Along $(1,-1,1)$: $\cdots DAE \sqcup DAE \sqcup DAE \sqcup DAE \cdots$
- Along $(1,1,-1)$: $\cdots CBE \sqcup CBE \sqcup CBE \sqcup CBE \cdots$

Concerning Type-I and Type-II, there is no gap along $\langle 111 \rangle$-directions. The fact brings Type-I and Type-II only the rhombus-type support.

3 Conclusion

An unknown ⟨110⟩ six-axes structure was reported. The parallel cylinders of the structure form a certain 2D rhombic lattice common to all of six directions. The six directions, A, B, $\cdots$, F, are classified into two groups α and β: A and B belong to α, and all the others to β. All the cylinders of group α are supported by contact with the cylinders of four directions of group β, so that the contacting four cylinders construct a rhombus. All the cylinders of group β are supported by cylinders of three directions: A, B and the perpendicular pair of itself, so that the three cylinders construct a equilateral triangle.

Researching of the derivative structures from the present structure and distinguishing the space group are further problems.

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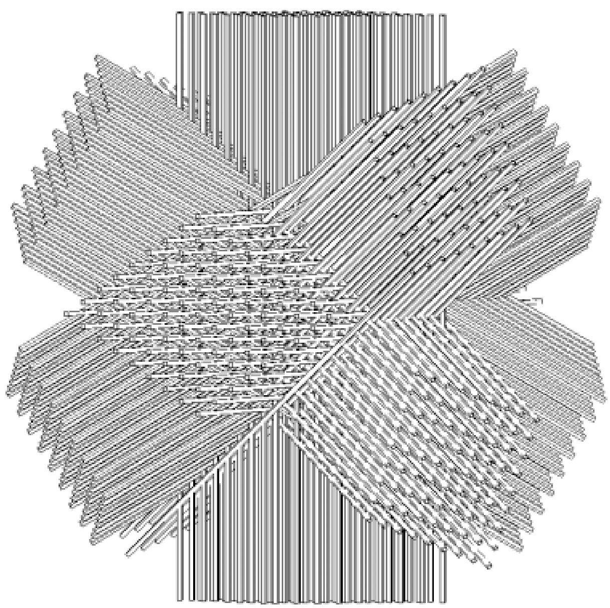

Fig. 1. An overview of the structure Type-III. Not only the present structure but also all ⟨110⟩-structures show the exterior of the rhombic dodecahedron if the models are made of many cylinders with the same length.

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Appendix: N-axes structures

Table 1. N-axes structures

N	Direction	Contact-Index†	Density	Figure
1	[001]	6^1	$\sqrt{3}\pi/6 \simeq 0.907$	Fig. 5
		4^1	$\pi/4 \simeq 0.785$	Fig. 6
2	[100], [010]	4^2	$\pi/4 \simeq 0.785$	Fig. 7
(k		4^k	$\pi/4 \simeq 0.785$	—)
3	$\langle 100 \rangle$	4^3	$3\pi/16 \simeq 0.589$	Fig. 8
4	$\langle 111 \rangle$	6^4	$\sqrt{3}\pi/8 \simeq 0.680$	Fig. 9
		3^4	$\sqrt{3}\pi/18 \simeq 0.302$	Fig. 10
6	$\langle 110 \rangle$	4^6	$\sqrt{2}\pi/9 \simeq 0.494$	Fig. 3
		5^6	$\sqrt{2}\pi/9 \simeq 0.494$	—
		4^6	$\sqrt{2}\pi/18 \simeq 0.247$	Fig. 4
		$4^4 5^2$	$\sqrt{2}\pi/18 \simeq 0.247$	—
		$4^2 5^4$	$\sqrt{2}\pi/18 \simeq 0.247$	—
		5^6	$\sqrt{2}\pi/18 \simeq 0.247$	—
		$3^4 4^2$	$\frac{(351\sqrt{2}-108\sqrt{6})\pi}{1936} \simeq 0.376$	Fig. 1, Fig. 2

$\langle 100 \rangle \equiv [100], [010], [001]$.
$\langle 111 \rangle \equiv [111], [1\bar{1}\bar{1}], [\bar{1}1\bar{1}], [\bar{1}\bar{1}1]$.
$\langle 110 \rangle \equiv [110], [101], [011], [1\bar{1}0], [\bar{1}01], [01\bar{1}]$.
†"Contact Index" shows the contact number with other cylinders in the each projection. A good example to understand the index is shown in Fig. 2

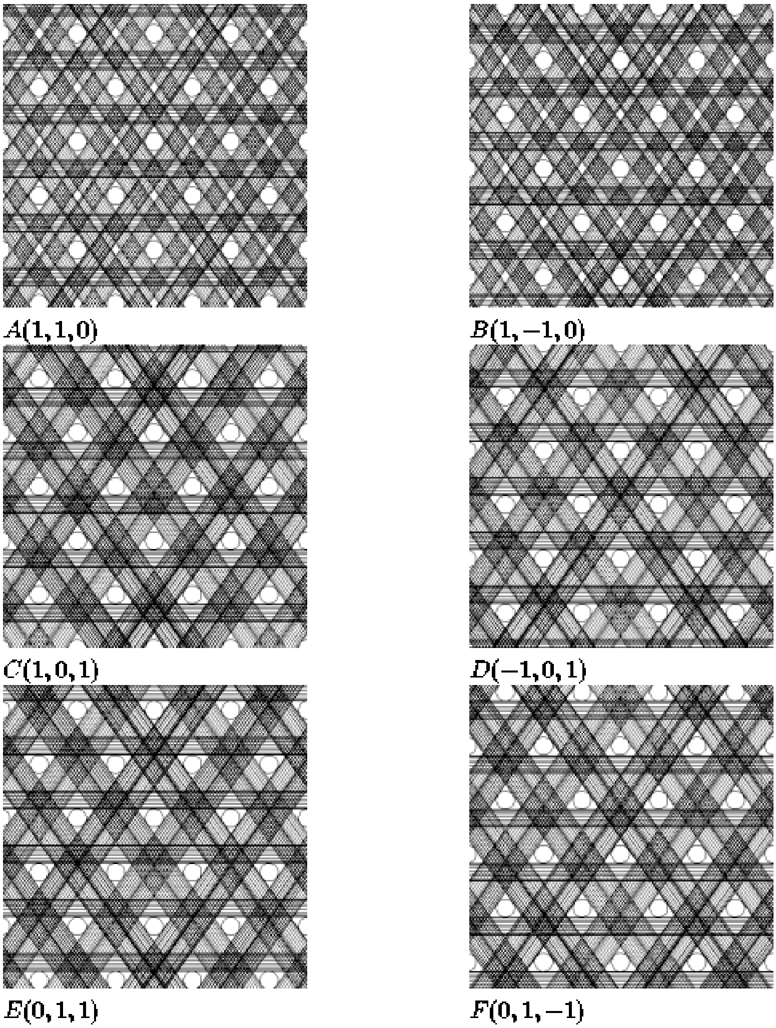
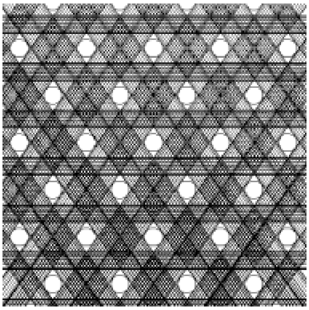
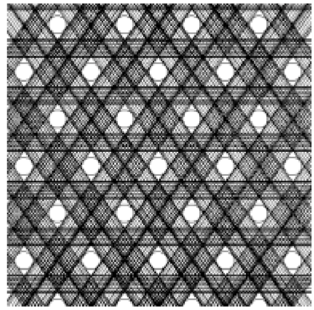


Fig. 2. Projection of the structure Type-III to each plane perpendicular to $\langle 110 \rangle$. A- and B-directions are fixed with the rhombic-type and other four directions are fixed with the equilateral-triangle-type. Thus, four directions are fixed by tri-gon, and two directions are fixed by tetra-gon, so we express the fixing state of the structure by $3^4 4^2$. (This is "Contact Index" in Table 1.) On these figures, upper direction is $+z$ for the projections of A and B, $+y$ for C and D, and $+x$ for E and F.

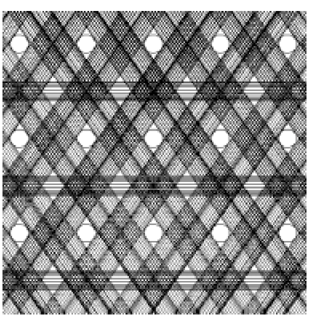


$A(1,1,0)$

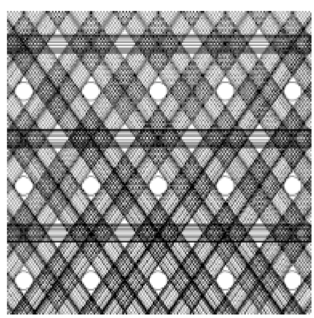


$B(1,-1,0)$

Fig. 3. Projection of the structure Type-I, whose packing density is $\sqrt{2}\pi/9 \simeq 0.494$. The figure is projections to A- and B-directions. The same projections as them are acquired for other four directions, therefore they are omitted. A cylinder is fixed with the rhombic-type in all directions, so the contact index is 4^6. Notice the existence of the structure with another contact index: 5^6 and with the same packing density. The fixing states described as another index contain the rhombic-type supporting.



$A(1,1,0)$



$B(1,-1,0)$

Fig. 4. Projection of the structure Type-II, whose packing density is $\sqrt{2}\pi/18 \simeq 0.247$. The figure is projections to A- and B-directions this time too, and other four are also omitted. Though the arrangement of cylinders in one direction is rectangular now, a cylinder is still fixed with the rhombic-type in all directions, so the contact index is 4^6. Notice the existence of the structures with other contact indexes: $4^4 5^2$,$4^2 5^4$ or 5^6 and with the same packing density. The fixing states described as the other indexes always contain the rhombic-type supporting.

Fig. 5. An overview of one-axis structure. Parallel cylinders form the honeycomb lattice. The packing density is $\sqrt{3}\pi/6 \simeq 0.907$. The contact index is 6^1. This structure is the densest of all cylinder packings

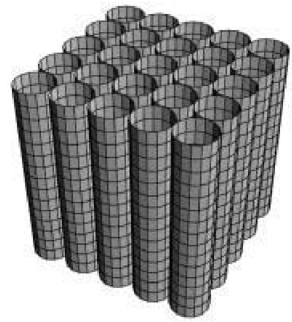


Fig. 6. An overview of another typical one-axis structure. Parallel cylinders form the square lattice. This is mechanically unstable. The packing density is $\pi/4 \simeq 0.785$, and the contact index is 4^1

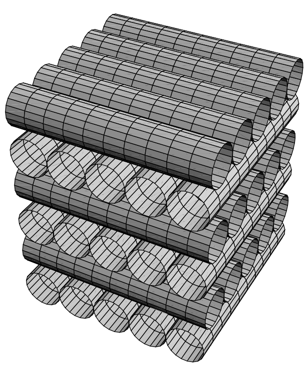


Fig. 7. Two-axes structure. The directions are [100] and [010] as typical ones. Arbitrary N-axes structures are possible if we extend this two-layer structure. Such N-axes structure have the same density and the same contact index as the two-axes structure. This structure is also unstable between different layer. The contact index is 4^2

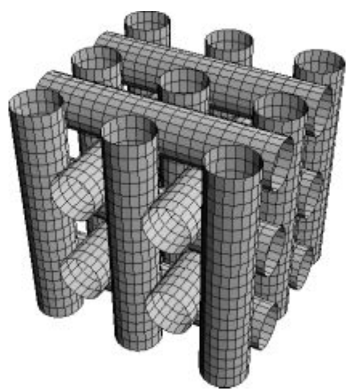


Fig. 8. Three-axes structure. The directions are $\langle 100 \rangle$. The packing density is $3\pi/16 \simeq 0.589$. The contact index is 4^3 which is the square-type supporting

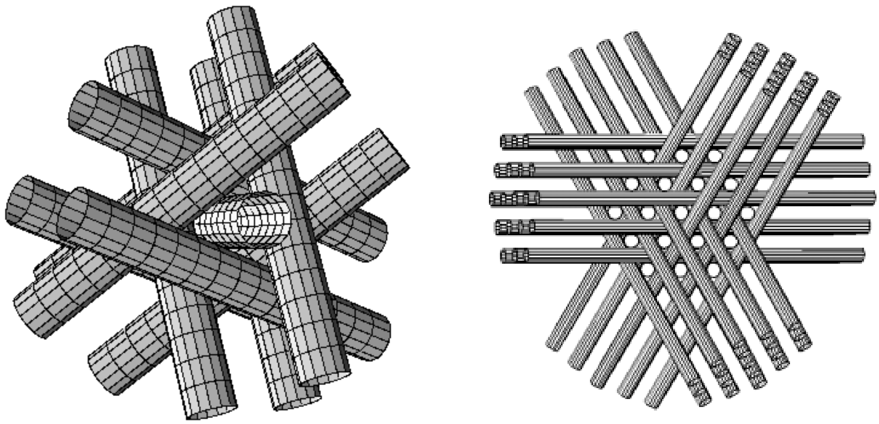


Fig. 9. Four-axes structure I(An overview and a projection). The directions are $\langle 111 \rangle$. The packing density is $\sqrt{3}\pi/8 \simeq 0.680$. The contact index is 6^4 which is the hexagonal-type supporting

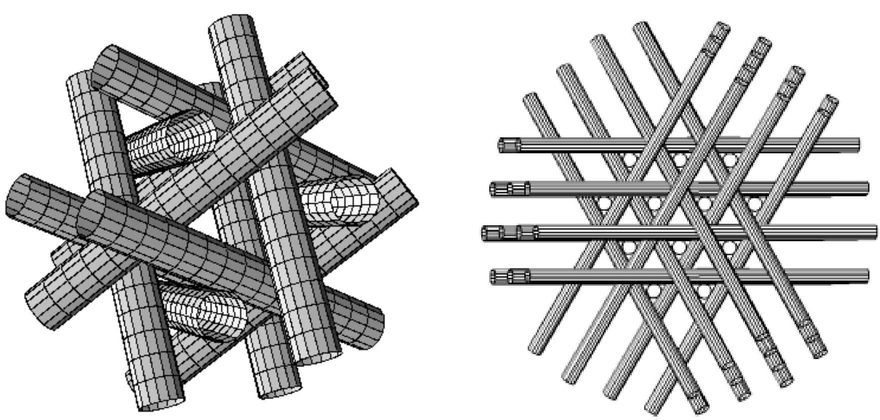


Fig. 10. Four-axes structure II(An overview and a projection). The directions are still $\langle 111 \rangle$ however the packing density is $\sqrt{3}\pi/18 \simeq 0.302$. and the contact index is 3^4. Two four-axes structures cannot come and go each other by any removal of cylinders

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B c ll we ve wo pp o c e o olve p oble O e e l d e o l p e c e c d e o e e do zed c e e l pp o c o olve P pe p oble

. S t n n t d n n p t n

ppo e e po p d p_2 e ve e we c co p e d p d $_w$ d p_2 b e e bo e c c e deed e e e bo e e po o p d e e e bo o p_2 $_2$ e d e e c be e l co p ed e o o e e wo d ce d d p p_2 e ce we co de e p p p_2

d d e o l p e e vec o e p oble b c ll p e c ve
o o e e e bo p oble Fo e c o e e e bo
d ce *d* p_i e ppe e velope c o o e cl de d
ce o $_i$ o p_i we co de c o e coo d e v l e
o p_i e cl de d ce co ve c o d ppe e velope o
co ve c o co ve c o e ce o objec ve c o w c
o ee co ve c o co ve c o
we c e l d e o l p e c e c e co ld e
ve lo e p oce o l o o e l o o e
d o de l o e dec o e de l o
e l e ce e p oble o d e o o o cell o e o o o
d oweve we e be e p llel l o b e e
bo ood q e d c e e ve b e de c p o e d c e
l o ol lo e l o l ec c l ec q e co p o l
eo e

Co de e o po *U* 2 e *d* d e o l p ce
w e e $_i$ $_i$ $_i$ d_i Fo po *d* e
q e o e d ce be wee d $_i$ *d* $_i^2$ $\sum_{j=}$ $_i^2$ I we
co de *f* $_i$ *d* $_i^2$ $\sum_{i=}$ ^{2} *f* $_i$ l e c o
po e e po o *p* o d o l
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e co de d l p oble e $_i$ be e pe pl e e *d* +
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e o o d o l e e p od c o ˜ d $_i$ e
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e *U* $_i$ Co de p e e *r*
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be o pl ce c b o *r* / o co Mo eove
e pe pl e o l po o e o e l p ce de ed b e
be o pl ce c b o *r* /⌊ /2 e d c e
co c ed lo *r* e

e c q e e po $_j$ z e e p od c w ve ˜
ollow e o e ve ce o *r* pl ce de ce d o de w
e pec o e e p od c w ˜ we ve o ed l o pl ce
cco d o e bove o ed o de o ve ce we co de e ve e
e l o *d* + ve ce o e c ple we c oo e e
r /⌊ /2 pl ce e l o pl ce Co de e po e *Y* ob
ed e o o $_i$ co e po d o e e pl ce e po $_j$
z e e p od c w ˜ be co ed *Y* B de o *Y*
co o *r* /⌊ /2 po e ce we c d $_j$ lo *r* p

llel ew $r+ r$ /⌊ /2 p oce o e e r /⌊ 3 /2 o b l ce r d r /⌊ /2 o e be o p oce o ⌊ 3 /2 e e l o o bo o de l o d dec o l o o e o p e c p oble o e o e op c ed ec ve e e bo e c d c e c e ed ce o ˜ ⌊ /2 w le e τ o be pol lo c

e co p o e o o p e c e c l o beco e lo + + r lo ^{2} r + lo

. *h i i di r ridg r d i* + lo *i*

e e co ple l l e o o e ol o b P pe p p o c ve e e b ec o oweve we c l ze pp o c l e o de e c e l o o e c e w e e we e ve e ve e ep e e o o e o

. S t n n t p p b

e P pe e od o e c e we e ve e ed ep e e o o e o e be e o o e e o ve ce o $_i$ d le be e e o l p ce de R_i e U

Fo e c b e U we co de bp oble de ed b e po d l p ce d le be e v l e o e objec ve c o d e e o e op l ol o o e bp oble e o e e p e e p ce we dd l p ce

Mo eove e ol o de ed b co be o e be o U I deed ppo e e po p_i o d $_i$ d e o l ce o R_i de ed b e bo d pe pl e o $_i$ l p ce d ve $_i$ e po $_i$ o e wo d o d $_i$ + d e o l ce o e e o o o d o $_i$ o de e e e e o d+ e be o e + + $_2$+ $_2$ e be o de e e bo p d p_2 q el e e o e d + e be c lled o o e ppe le d + e be de e e p d p_2 o e ple e d e e p le wo be d

pop l do zed e o l o o olv e P pe p ob le wo I e ze o o d + we d ec l olve e p oble b e c e ce we ve ed e d e o d ed co e b e c e c be olved e

I ew po w o lo o e e l we e e ed o we c ec w e e e d ce d p o e c e b d po p l e e d ce be wee p d po o I o l e I l e we pd e e b e b ec vel c ll e l o w e e we l ze e o o e ew po be e be o I o e wo d e ec ve co c o elec ed d e ele e o e do l e ed

I clo ed l pl e e de e o R e ed we c ec
w e e e c e b d po p loc ed o o I loc ed
O e w e we pd e e b e ec vel c ll
e l o w e e we l ze e o o e bo d pe pl e
o be e be o

B e b c w d l o do zed l o e p oble
c be olved + do zed e pec ed e w e e e o
be o ve ce $_i$ k d $\sum_i$ $_i$

. *h i i di r ridg r d i* + *r d i d d i*

. t x p nt t n n

e wo ld l e o co de e c e w e e e e o R_i e ve e ve e
ep e e o o R_i $\sum$ $\sum$
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e c o l e e p oble o d e e P pe p oble w e e we
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de e ce w o lo o e e l we c e e ce
d d e o l ple $_i$ co p_i o e c

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.5 d p nt t n n

e wo ld l e o co de e c e w e e e e o R_i ve e ec
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e $R \times R_2$ be e e ble e o w c we e co de e e
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e po f_i R_i o p_i o e c o o e
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p t n . *h d i i h r d i d g*
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e e e eve l e e l z o o e d e e b d e p oble I
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J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 370-380, 2001.

P a e em

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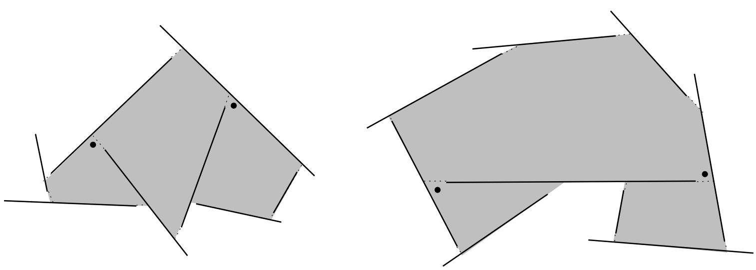

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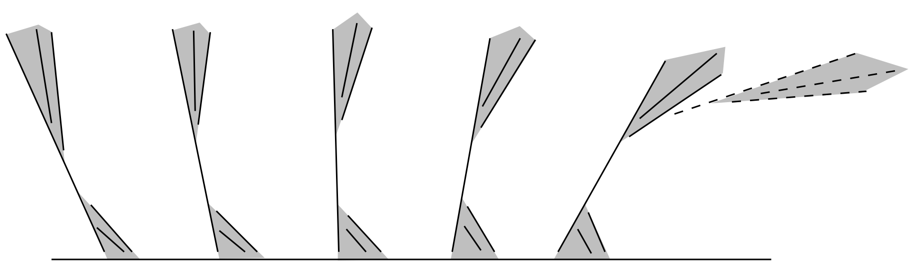

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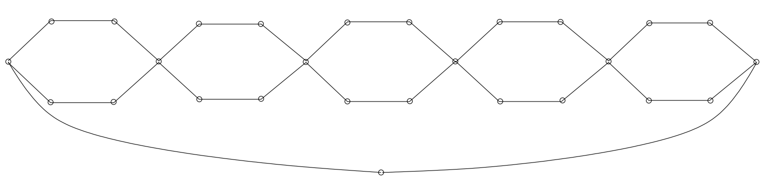

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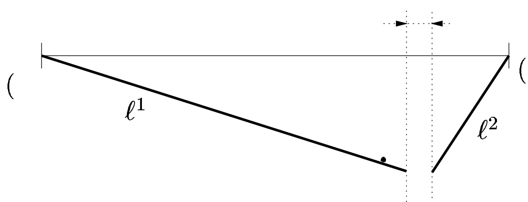

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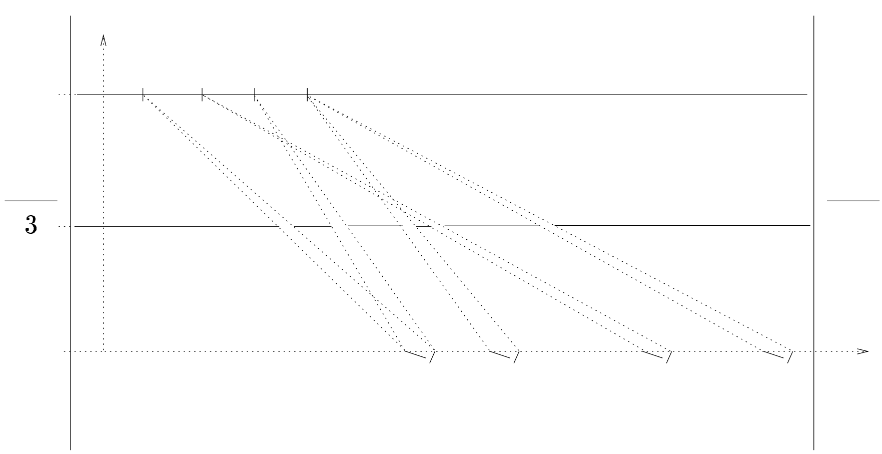

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A e me s

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